

On the Symmetric Feedback Capacity of the K -user Cyclic Z-Interference Channel*

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Abstract

The K -user cyclic Z-interference channel models a situation in which the k th transmitter causes interference only to the $(k - 1)$ th receiver in a cyclic manner, i.e., the first transmitter causes interference only to the K th receiver. The impact of noiseless feedback on the capacity of this channel is studied by focusing on the Gaussian cyclic Z-interference channel. To this end, the symmetric feedback capacity of the linear shift deterministic cyclic Z-interference channel (LD-CZIC) is completely characterized for all interference regimes. Using insights from the linear deterministic channel model, the symmetric feedback capacity of the Gaussian cyclic Z-interference channel is characterized up to within a constant number of bits. As a byproduct of the constant gap result, the symmetric degrees of freedom with feedback for the Gaussian cyclic Z-interference channel are also characterized. These results highlight that the symmetric feedback capacities for both linear and Gaussian channel models are in general functions of K , the number of users. Furthermore, the capacity gain obtained due to feedback decreases as K increases.

1 Introduction

Managing the effects of interference is a key issue in currently deployed wireless networks. Among several ways to mitigate or perhaps constructively using interference is to make use of cooperation amongst interfering users. In this paper, we focus on one such important issue by studying the impact of noiseless receiver-to-transmitter feedback on the capacity

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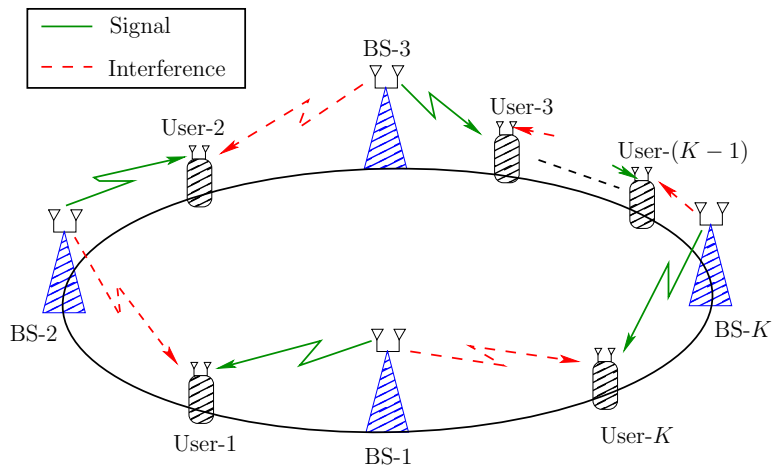


Figure 1: Modified Wyner Model.

of the K -user cyclic Z-interference channel (CZIC). In this model, K transmitters intend to transmit independent messages to K respective receivers and the k th transmitter causes interference to the $(k - 1)$ th receiver in a cyclic manner. The motivation for studying the cyclic Z-interference channel comes from the modified Wyner model [1], which describes the soft handoff scenario of a cellular network. In the original Wyner model [2], each receiver can suffer interference from its adjacent transmitters. In the modified Wyner model, one can assume that the terminals are situated along a circular array (see Figure 1). If in addition, we assume that the mobile communicates with the intended base-station on its left (or right), while suffering interference due to the BS to its right (or left), then the resulting channel model is the K -user CZIC, which is considered in this paper. The K -user Gaussian CZIC (G-CZIC) *without feedback* was recently investigated in [3], where it was shown that the generalized degrees-of-freedom of the symmetric K -user G-CZIC are the same as for the 2-user Gaussian interference channel. By an interesting generalization of the results of Etkin, Tse and Wang [4], the approximate symmetric capacity was characterized for the weak interference regime and the exact capacity region was characterized for the strong interference regime. A simpler variation of the Gaussian K -user CZIC has been studied in [5], where the results of [3] are strengthened for the 3-user case. It is shown in [5] that a generalization of the Han-Kobayashi [6] scheme can achieve sum-capacity for some interference regimes.

In this paper we focus on the K -user CZIC *with feedback*, i.e., we assume the presence of noiseless and causal feedback from the k th receiver to the k th transmitter. For $K = 2$, this model reduces to the conventional 2-user interference channel with feedback. For $K > 2$, this model is a special case of the general K -user interference channel with feedback (see Figure 2). The 2-user interference channel with various forms of feedback has been investigated recently. Feedback coding schemes for K -user Gaussian interference networks have been developed by Kramer in [7]. Outer bounds for the 2-user interference channel with generalized feedback have been derived in [8] and [9](also see references therein). The 2-user Gaussian interference

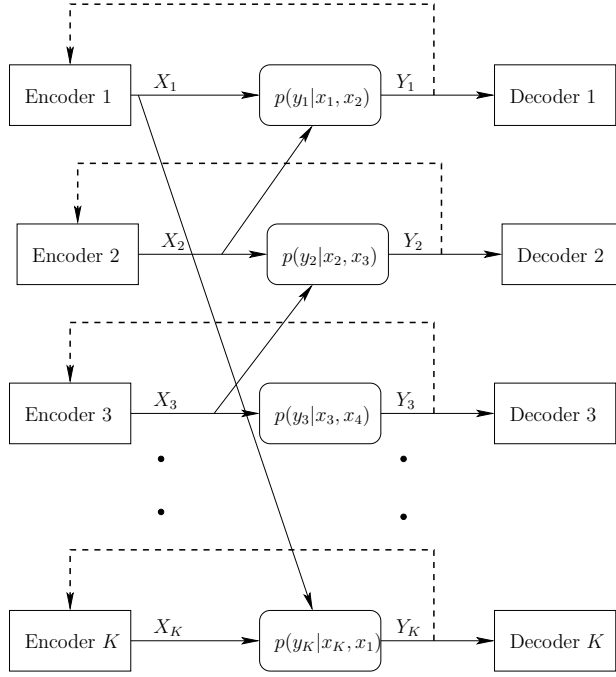


Figure 2: K-user cyclic Z-interference channel with feedback.

channel with noiseless (channel output) feedback was considered in [10] and the feedback capacity region was characterized to within two bits. One of the main findings in [10] is that feedback provides multiplicative gain at high signal-to-noise ratio (SNR) and the gain becomes arbitrarily large for certain channel parameters. The key insights that led to this result were obtained by characterizing the feedback capacity region of the linear deterministic (LD) 2-user interference channel. The linear deterministic model despite its simplicity can provide valuable insights for the Gaussian channel model.

With this correspondence at hand, we first focus on the linear deterministic K -user CZIC with feedback. We characterize the symmetric feedback capacity, $\mathcal{C}_{\text{sym,LD}}^{\text{FB}}$, which is defined as the maximum R such that the rate K -tuple (R, R, \dots, R) is achievable with feedback. We use insights from the linear deterministic model to characterize the symmetric feedback capacity of the K -user Gaussian CZIC within a constant number of bits (independent of the channel gains) for all interference regimes. As a consequence of our constant gap results, we also establish the generalized feedback degrees of freedom of the Gaussian CZIC. For the scope of this paper, we restrict our attention to the case of symmetric channel parameters. For instance, for the Gaussian CZIC with feedback, we assume that the direct channel gain from the k th transmitter to the k th receiver is same for all k , and that the interference channel gain from the k th transmitter to the $(k - 1) \bmod (K)$ th receiver is the same for all k .

The symmetric K -user Gaussian CZIC is described by the pair (SNR, INR) , where SNR denotes the direct channel gain and INR denotes the interference channel gain. The degrees

of freedom (**DoF**) of the Gaussian CZIC (per-user) *without feedback* is defined as

$$\mathbf{DoF}(\alpha, K) = \frac{1}{K} \lim_{\text{SNR} \rightarrow \infty} \frac{\mathcal{C}_{\text{sum,G}}(K)}{\frac{1}{2} \log(1 + \text{SNR})}, \quad (1)$$

where $\mathcal{C}_{\text{sum,G}}(K)$ is the sum-capacity *without feedback*, and α is the interference parameter, defined as $\alpha \triangleq \frac{\log(\text{INR})}{\log(\text{SNR})}$. Analogous to (1), we define the generalized feedback degrees of freedom of the Gaussian CZIC (per-user) as

$$\mathbf{DoF}^{\text{FB}}(\alpha, K) = \frac{1}{K} \lim_{\text{SNR} \rightarrow \infty} \frac{\mathcal{C}_{\text{sum,G}}^{\text{FB}}(K)}{\frac{1}{2} \log(1 + \text{SNR})}, \quad (2)$$

where $\mathcal{C}_{\text{sum,G}}^{\text{FB}}(K)$ is the sum-capacity *with feedback*.

In the breakthrough paper [4], several novel results were obtained for 2-user Gaussian interference channel. Among one of the results, is the characterization of the **DoF**. In particular, it was shown that $\mathbf{DoF}(\alpha, 2) = \min(\max(1 - \alpha, \alpha), 1 - \alpha/2)$. On the other hand, the feedback **DoF** for $K = 2$ was recently characterized in [10] and is given as $\mathbf{DoF}^{\text{FB}}(\alpha, 2) = \max(1 - \alpha/2, \alpha/2)$. From the results of [3], it is clear that the **DoF** without feedback for the K -user Gaussian CZIC is the same for all $K \geq 2$, i.e., it is *independent* of K , the number of users. It is natural to ask the question that this equivalence continues to hold in presence of feedback for $K > 2$?

We answer this question in the negative by showing that the feedback **DoF** of the K -user Gaussian CZIC is in general a function of K . Of particular interest is the very strong interference regime, corresponding to $\alpha \geq 2$. In this regime the feedback **DoF** for the 2-user case is given as $\mathbf{DoF}^{\text{FB}}(\alpha, 2) = \alpha/2$. This implies that the feedback gain can be unbounded as α increases. For this regime, we show that the feedback **DoF** of the K -user Gaussian CZIC is given as $\mathbf{DoF}^{\text{FB}}(\alpha, K) = 1 + \frac{(\alpha-2)}{K}$. This result shows that for a fixed α , as the number of users increase, the feedback gain decreases and completely vanishes in the limit $K \rightarrow \infty$. The outer bounds derived in this paper to establish capacity/constant bit gap results can be regarded as genie aided bounds derived for the 2-user case considered in [10]. However, as K , the number of users increases, selecting appropriate genies becomes prohibitively complex. In particular, for the K -user CZIC, we have a total of $K!$ sum-rate upper bounds. Depending on the interference parameter α , we carefully select the best upper bound among the $K!$ upper bounds.

This paper is organized as follows. In Section 2, we describe the K -user cyclic Z-interference channel with feedback. In Section 3 we describe our main results for both linear deterministic and Gaussian K -user CZICs. We provide the intuition as to why the feedback gain decreases as the number of users increases. Proofs for the K -user linear deterministic CZIC are presented in Sections 4 and 5. Constant gap results for the feedback capacity of the K -user Gaussian CZIC are established in Section 6. We conclude the paper in Section 7.

2 K -user Cyclic Z-Interference Channel with Feedback

The K -user cyclic Z-interference channel is described by K conditional probabilities $\{p(y_1|x_1, x_2), p(y_2|x_2, x_3), \dots, p(y_K|x_K, x_1)\}$. A (T, M_1, \dots, M_K) feedback code for the CZIC consists of sequences of K encoding functions

$$f_{k,t} : \{1, \dots, M_k\} \times \mathcal{Y}_k^{t-1} \rightarrow \mathcal{X}_{k,t}, \quad k = 1, \dots, K, \quad (3)$$

for $t = 1, \dots, T$, and K decoding functions

$$g_k : \mathcal{Y}_k^T \rightarrow \{1, \dots, M_k\}, \quad k = 1, \dots, K. \quad (4)$$

The probability of decoding error at decoder k is denoted by P_k and is defined as $P_k = \mathbb{P}(g_k(Y_k^T) \neq W_k)$, where W_k is the message of transmitter k .

A rate K -tuple (R_1, \dots, R_K) is achievable for the K -user CZIC if there exists a (T, M_1, \dots, M_K) feedback code such that $\log(M_k)/T \leq R_k - \epsilon_{k,T}$ and $P_k \leq \epsilon_{k,T}$, where $\epsilon_{k,T} \rightarrow 0$ as $T \rightarrow \infty$ for all k . The feedback capacity region $\mathcal{C}^{\text{FB}}(K)$ is the set of all achievable K -tuples.

In this paper, we focus on the symmetric feedback capacity of the K -user CZIC, denoted by $\mathcal{C}_{\text{sym}}^{\text{FB}}(K)$, which is defined as the maximum R such that $(R, \dots, R) \in \mathcal{C}^{\text{FB}}(K)$.

2.1 Linear deterministic CZIC with Feedback

The symmetric linear deterministic CZIC is described by a pair of integers (n, m) , where n denotes the number of signal (direct) levels and m denotes the number of interference levels observed at the receivers.

The channel input of transmitter k , denoted by X_k , for $k = 1, \dots, K$, is assumed to be of length $\max(n, m)$.

When $n \geq m$, we denote

$$\begin{aligned} U_k &: \text{top-most } (n - m) \text{ bits of } X_k \\ V_k &: \text{top-most } m \text{ bits of } X_k \\ L_k &: \text{lower-most } m \text{ bits of } X_k. \end{aligned} \quad (5)$$

With this notation, we can write the channel outputs for the K -user LD-CZIC as follows:

$$Y_k = (U_k, L_k \oplus V_{k+1}), \quad (6)$$

for $k = 1, \dots, K$.

When $n < m$, we denote

$$\begin{aligned} U_k &: \text{top-most } (m - n) \text{ bits of } X_k \\ V_k &: \text{top-most } n \text{ bits of } X_k \\ L_k &: \text{lower-most } n \text{ bits of } X_k. \end{aligned} \tag{7}$$

With this notation, we can write the channel outputs for the K -user LD-CZIC as follows:

$$Y_k = (U_{k+1}, L_{k+1} \oplus V_k), \tag{8}$$

for $k = 1, \dots, K$, where we define

$$V_{K+1} \triangleq V_1, \quad U_{K+1} \triangleq U_1, \quad L_{K+1} \triangleq L_1 \tag{9}$$

for consistency.

For instance, when $n \geq m$, the 3-user LD-CZIC is described by the following input-output relationships:

$$\begin{aligned} Y_1 &= (U_1, L_1 \oplus V_2) \\ Y_2 &= (U_2, L_2 \oplus V_3) \\ Y_3 &= (U_3, L_3 \oplus V_1). \end{aligned}$$

2.2 Gaussian K -user CZIC with Feedback

To describe the K -user Gaussian CZIC, we denote¹ the signal transmitted by user k as X_k . We impose an average unit power constraint at each user; that is $\mathbb{E}[X_k^2] \leq 1$. The signal observed at receiver k is obtained by

$$Y_k = \sqrt{\text{SNR}}X_k + \sqrt{\text{INR}}X_{k+1} + Z_k, \quad k = 1, 2, \dots, K, \tag{10}$$

where we define $X_{K+1} \triangleq X_1$ for consistency, and the noise Z_k at receiver k is zero mean Gaussian random variable with unit variance. Moreover, the noises across the receivers are assumed to be independent, i.e., Z_k and $Z_{k'}$ are independent for $k \neq k'$.

3 Main Results

The results for the linear deterministic model are presented in terms of the interference parameter α , which is defined in this model as the ratio of the number of interference levels

¹With slight abuse of notation, we use similar notation for both LD-CZIC and G-CZIC channel models. However, the corresponding notation should be clear from the context.

to the number of signal levels, i.e.,

$$\alpha \triangleq \frac{m}{n}. \quad (11)$$

We define the normalized² symmetric feedback capacity *per-user* of the LD-CZIC as follows:

$$\mathcal{C}_{\text{sym,LD}}^{\text{FB}}(\alpha, K) \triangleq \frac{1}{K} \frac{\mathcal{C}_{\text{sum,LD}}^{\text{FB}}(K)}{n}, \quad (12)$$

where $\mathcal{C}_{\text{sum,LD}}^{\text{FB}}(K)$ is the feedback sum-capacity of the K -user LD-CZIC.

We present our first result in the following theorem:

Theorem 1 *The normalized symmetric feedback capacity, $\mathcal{C}_{\text{sym,LD}}^{\text{FB}}(\alpha, K)$ of the K -user LD-CZIC is given as*

$$\mathcal{C}_{\text{sym,LD}}^{\text{FB}}(\alpha, K) = \begin{cases} (1 - \alpha) + \frac{\alpha}{K}, & 0 \leq \alpha \leq 1/2 \\ \alpha + \frac{(2-3\alpha)}{K}, & 1/2 \leq \alpha \leq 2/3 \\ 1 - \frac{\alpha}{2}, & 2/3 \leq \alpha \leq 1 \\ \frac{\alpha}{2}, & 1 \leq \alpha \leq 2 \\ 1 + \frac{(\alpha-2)}{K}, & \alpha \geq 2, \end{cases} \quad (13)$$

Theorem 1 is proved in two parts: feedback coding schemes are presented in Section 4 and corresponding upper bounds for the normalized symmetric feedback capacity are obtained in Section 5.

The constant bit gap results for the Gaussian model are presented in terms of two parameters $(C_{\text{SNR}}, C_{\text{INR}})$, defined as:

$$C_{\text{SNR}} \triangleq \frac{1}{2} \log(1 + \text{SNR}) \quad (14)$$

$$C_{\text{INR}} \triangleq \frac{1}{2} \log(1 + \text{INR}). \quad (15)$$

We also define the interference parameter for the Gaussian model as

$$\alpha = \frac{\log(\text{INR})}{\log(\text{SNR})}. \quad (16)$$

We define the symmetric feedback capacity *per-user* of the K -user Gaussian CZIC as follows:

$$\mathcal{C}_{\text{sym,G}}^{\text{FB}}(K) \triangleq \frac{\mathcal{C}_{\text{sum,G}}^{\text{FB}}(K)}{K}, \quad (17)$$

²normalized with respect to the number of direct levels, n .

where $\mathcal{C}_{\text{sum,G}}^{\text{FB}}(K)$ is the feedback sum-capacity of the K -user Gaussian CZIC. We next define the feedback degrees of freedom *per-user* for the K -user Gaussian CZIC as follows:

$$\mathbf{DoF}^{\text{FB}}(\alpha, K) = \lim_{\text{SNR} \rightarrow \infty} \frac{\mathcal{C}_{\text{sym,G}}^{\text{FB}}(K)}{\frac{1}{2} \log(1 + \text{SNR})}. \quad (18)$$

We present our next result in the following theorem:

Theorem 2 *The symmetric feedback capacity per user, $\mathcal{C}_{\text{sym,G}}^{\text{FB}}(K)$ of the K -user Gaussian CZIC satisfies*

$$\mathcal{C}_{\text{sym,G}}^{\text{FB}}(K) \simeq \begin{cases} C_{\text{SNR}} - C_{\text{INR}} + \frac{C_{\text{INR}}}{K}, & 0 \leq \alpha \leq 1/2 \\ C_{\text{INR}} + \frac{(2C_{\text{SNR}} - 3C_{\text{INR}})}{K}, & 1/2 \leq \alpha \leq 2/3 \\ C_{\text{SNR}} - \frac{C_{\text{INR}}}{2}, & 2/3 \leq \alpha \leq 1 \\ \frac{C_{\text{INR}}}{2}, & 1 \leq \alpha \leq 2 \\ C_{\text{SNR}} + \frac{(C_{\text{SNR}} - 2C_{\text{INR}})}{K}, & \alpha \geq 2, \end{cases} \quad (19)$$

where the notation $A \simeq B$ implies that $(A - B) \leq 3$, i.e., the worst case gap (for all interference regimes) between the upper and lower bounds is at most 3 bits/user.

Theorem 2 is proved in Section 6, where we use key insights from the linear deterministic model to construct feedback coding schemes and corresponding upper bounds on the feedback sum capacity. Further analysis of these bounds shows that they differ by a constant number of bits, which is independent of (SNR, INR) . We note here that the worst case gap of 3 can be reduced depending on the interference regime. For instance, in our proof of Theorem 2, we show that the gap for the case when $\alpha > 2$ is at most 2 bits.

As a consequence of Theorem 2, we have the following corollary:

Corollary 1 *The feedback degrees of freedom per-user of the K -user Gaussian CZIC is given as*

$$\mathbf{DoF}^{\text{FB}}(\alpha, K) = \begin{cases} (1 - \alpha) + \frac{\alpha}{K}, & 0 \leq \alpha \leq 1/2; \\ \alpha + \frac{(2-3\alpha)}{K}, & 1/2 \leq \alpha \leq 2/3; \\ 1 - \frac{\alpha}{2}, & 2/3 \leq \alpha \leq 1; \\ \frac{\alpha}{2}, & 1 \leq \alpha \leq 2; \\ 1 + \frac{(\alpha-2)}{K}, & \alpha \geq 2. \end{cases} \quad (20)$$

We recall the no-feedback degrees of freedom per-user of the K -user Gaussian CZIC [3]:

$$\mathbf{DoF}(\alpha, K) = \begin{cases} (1 - \alpha), & 0 \leq \alpha \leq 1/2 \\ \alpha, & 1/2 \leq \alpha \leq 2/3 \\ 1 - \frac{\alpha}{2}, & 2/3 \leq \alpha \leq 1 \\ \frac{\alpha}{2}, & 1 \leq \alpha \leq 2 \\ 1, & 2 \leq \alpha. \end{cases} \quad (21)$$

Note that $\mathbf{DoF}(\alpha, K)$ is *independent* of K , i.e. $\mathbf{DoF}(\alpha, K) = \mathbf{DoF}(\alpha, 2)$, for all K . This implies that from the \mathbf{DoF} point of view, the behavior of the K -user system is similar to the $K = 2$ user system in the *absence* of feedback.

On the other hand, we note that the feedback \mathbf{DoF} for $K = 2$ is given as [10]

$$\mathbf{DoF}^{\text{FB}}(\alpha, 2) = \begin{cases} 1 - \frac{\alpha}{2}, & \alpha \leq 1 \\ \frac{\alpha}{2}, & \alpha \geq 1. \end{cases} \quad (22)$$

In the light of above observations, it is natural to ask whether the behavior of the K -user Gaussian CZIC mimics the behavior of the $K = 2$ system in the presence of feedback. Corollary 1 answers this question in the negative by showing that the feedback \mathbf{DoF} per-user for $K > 2$ is in general a function of K . Moreover, the feedback \mathbf{DoF} of $K = 2$ always serves as an upper bound for the feedback \mathbf{DoF} for $K > 2$ users.

In Figure 3, the feedback \mathbf{DoF} s are shown for the K -user Gaussian CZIC, when $K = 2, 4$ and 10.

Remark 1 *Corollary 1 also shows that $\mathbf{DoF}^{\text{FB}}(\alpha, K)$ can be strictly less than $\mathbf{DoF}^{\text{FB}}(\alpha, 2)$ (see Figure 3). Secondly, it also shows that $\mathbf{DoF}^{\text{FB}}(\alpha, K)$ is monotonically decreasing in K . Hence, as the number of users in the system increase, the \mathbf{DoF} gain obtained via feedback decreases. Furthermore, in the limit $K \rightarrow \infty$, the feedback gain vanishes, i.e., we have*

$$\lim_{K \rightarrow \infty} \mathbf{DoF}^{\text{FB}}(\alpha, K) = \mathbf{DoF}(\alpha, K). \quad (23)$$

4 Feedback Coding Schemes for LD-CZIC

4.1 Very-weak interference: $0 \leq \alpha \leq 1/2$

In this regime, we show that $K(n - m) + m$ bits per user can be reliably sent in K channel uses.

As an example, we start with the case in which $K = 4$ and $m = 1, n = 3$, so that $\alpha = 1/3$. To achieve 9 bits per user in 4 channel uses, the following coding scheme is used (see Figure 4):

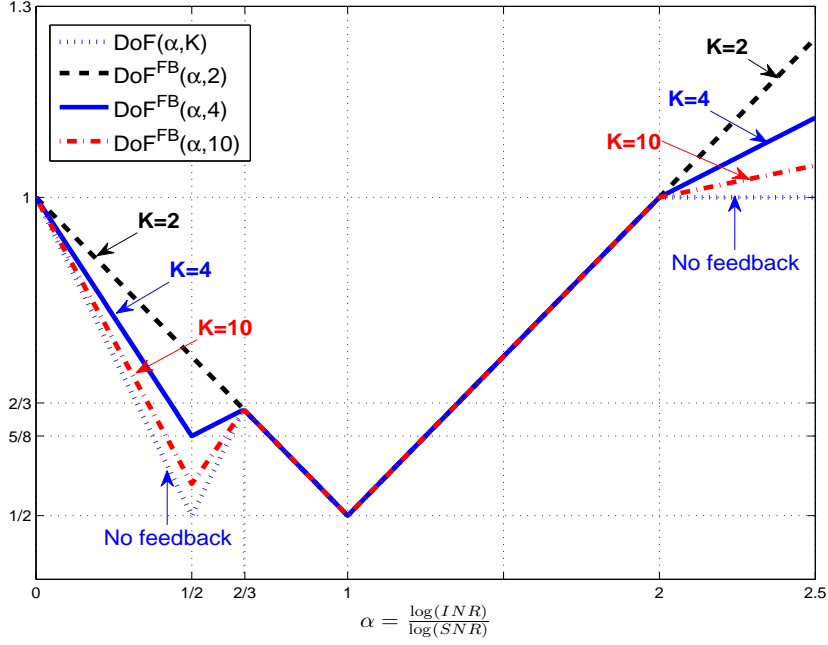


Figure 3: Feedback **DoF** of the K -user Gaussian CZIC.

- In the first channel use, each encoder transmits fresh bits on all levels (for example, encoder 1 sends a_1, a_2, a_3).
- Upon receiving feedback, each encoder can decode the upper most bit of the next encoder (encoder 1 decodes b_1 , encoder 2 decodes c_1 , etc.).
- In all subsequent channel uses, each encoder transmits the previously decoded bit on the top most level and fresh information bits in the remaining lower two levels (at $t = 2$ encoder 1 transmits b_1 on the top level and a_4, a_5 on the two lower levels).
- From Figure 4 it is clear that each user can reliably transmit 9 bits to its decoder in 4 channel uses. Hence, this scheme yields a normalized symmetric rate of $(9/4) * (1/3) = 3/4$.

This scheme can be readily generalized for *arbitrary* numbers of users K and for any $\alpha \in [0, 1/2]$ as follows: at $t = 1$, every encoder transmits n fresh information bits. Using feedback, the k th encoder decodes the lower most m bits transmitted by the $(k + 1)$ th encoder. For all subsequent $1 < t \leq K$, the k th encoder transmits the previously decoded m bits on the top most m levels and transmits fresh information in the lower $(n - m)$ levels. This scheme achieves $K(n - m) + m$ bits per user in K channel uses and the achievable rate is $(n - m) + m/K$. Hence for $\alpha \in [0, 1/2]$, we have

$$\mathcal{C}_{\text{sym,LD}}^{\text{FB}}(\alpha, K) \geq (1 - \alpha) + \frac{\alpha}{K}. \quad (24)$$

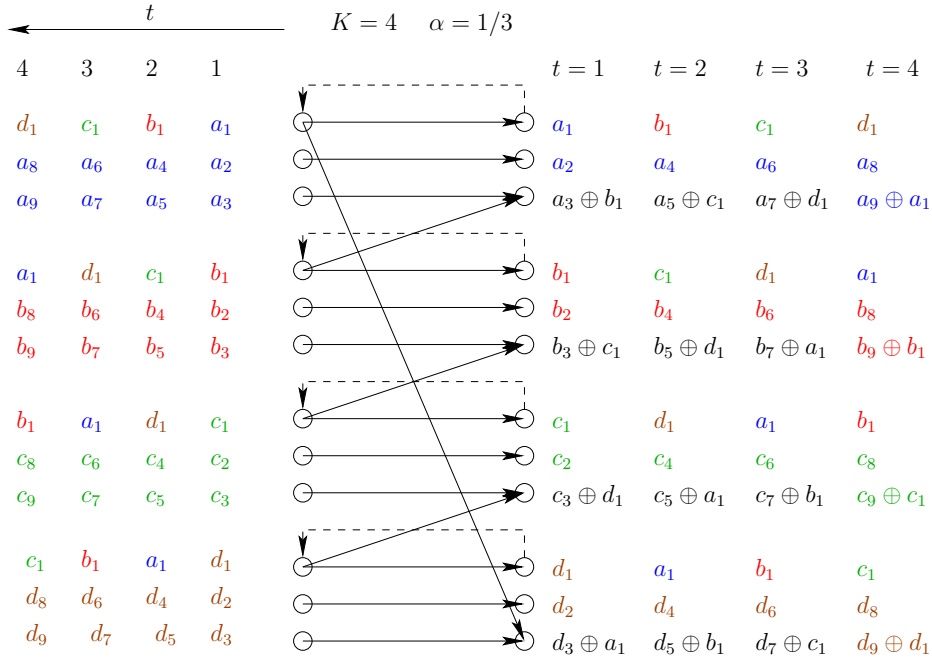


Figure 4: Feedback Coding scheme for $\alpha = 1/3$, $K = 4$.

4.2 Weak interference: $1/2 \leq \alpha \leq 2/3$

For this regime, we present a feedback coding scheme that achieves $Km + (2n - 3m)$ bits per user in K channel uses. We break the channel input of encoder k into four mutually exclusive sets of levels as follows: $X_k(t) = (X_{k,1}(t), X_{k,2}(t), X_{k,3}(t), X_{k,4}(t))$, where the number of bits in $X_{k,r}(t)$ are $(2m - n)$, $(2n - 3m)$, $(2m - n)$ and $(n - m)$, for $r = 1, 2, 3$ and 4, respectively.

At $t = 1$, each encoder transmits fresh information bits on $X_{k,1}(1)$, $X_{k,2}(1)$ and $X_{k,4}(1)$ levels. For all $t \in \{1, \dots, K\}$, all encoders remains silent in the $X_{k,3}(t)$ level. At t , due to feedback, encoder k can decode the bits transmitted by the encoder $(k + 1)$ in the second subset level, i.e., it can decode $X_{(k+1),2}(t)$. For all $1 < t \leq K$, encoder k transmits

$$X_k(t) = (X_{k,1}(t), X_{(k+1),2}(t-1), \phi, X_{k,4}(t)),$$

where $X_{k,1}(t)$ and $X_{k,4}(t)$ consist of fresh information bits. It is clear that Km bits are achievable from the levels 1 and 4. A gain of $(2n - 3m)$ bits is provided by feedback in K usages of the channel. It can be easily verified that this coding scheme yields $Km + (2n - 3m)$ bits per user in K channel uses. Hence, we have

$$\mathcal{C}_{\text{sym,LD}}^{\text{FB}}(\alpha, K) \geq \alpha + \frac{(2 - 3\alpha)}{K}. \quad (25)$$

4.3 Moderate-strong interference: $2/3 \leq \alpha \leq 2$

In this regime, Theorem 1 shows that feedback does not increase the normalized symmetric capacity and hence the no-feedback coding scheme in [3] suffices.

4.4 Very-strong interference: $\alpha \geq 2$

In this regime, we will show that $(K - 2)n + m$ bits per user are achievable in K channel uses.

As an example, we start with the case in which $K = 4$ and $m = 3$ and $n = 1$, so that $\alpha = 3$. To achieve 5 bits per user in 4 channel uses, the following coding scheme is used (see Figure 5):

- In all channel uses, each encoder remains silent in the lower most bit. In the first channel use, each encoder transmits 2 fresh bits (for instance, encoder 1 sends a_1, a_2 and encoder 2 sends b_1, b_2). Using feedback, each encoder can decode the second bit transmitted by the encoder interfering its decoder (encoder 1 decodes b_2 , encoder 2 decodes c_2 etc.).
- In all subsequent channel uses, each encoder transmits a fresh information bit in the top-most level and the previously decoded bit in the second level (for instance, at $t = 2$, encoder 2 sends the fresh bit b_3 in the top-most level and the decoded bit c_2 in the second level).
- From Figure 5, it is clear that in 4 channel uses, using the top most level, each decoder receives 4 bits. One more bit is received from the interfering user in the final channel use (for instance, the bit a_2 is received at decoder 1 in a delayed manner). This scheme yields a rate of $5/4$ per user.

This scheme can be readily generalized for *arbitrary* numbers of users K and for any $\alpha \geq 2$ as follows: for any $1 \leq t \leq K$, all encoders do not transmit any information in the lower most n levels. At $t = 1$, the k th encoder transmits $(m - n)$ fresh information bits in the top $(m - n)$ levels. Using feedback, it decodes the $(m - n)$ bits transmitted by the $(k + 1)$ th encoder. For any $1 < t \leq K$, the k th encoder transmits fresh information on the top most n levels and in the remaining $(m - 2n)$ levels, it transmits the lower $(m - 2n)$ bits decoded at $(t - 1)$. This scheme achieves $Kn + (m - 2n)$ bits per user in K channel uses. Hence for $\alpha \geq 2$, we have

$$\mathcal{C}_{\text{sym,LD}}^{\text{FB}}(\alpha, K) \geq 1 + \frac{(\alpha - 2)}{K}. \quad (26)$$

Theorem 1 shows that the feedback coding scheme presented above is optimal. Therefore, it is clear that the gain obtained via feedback should decrease as the number of users increases.

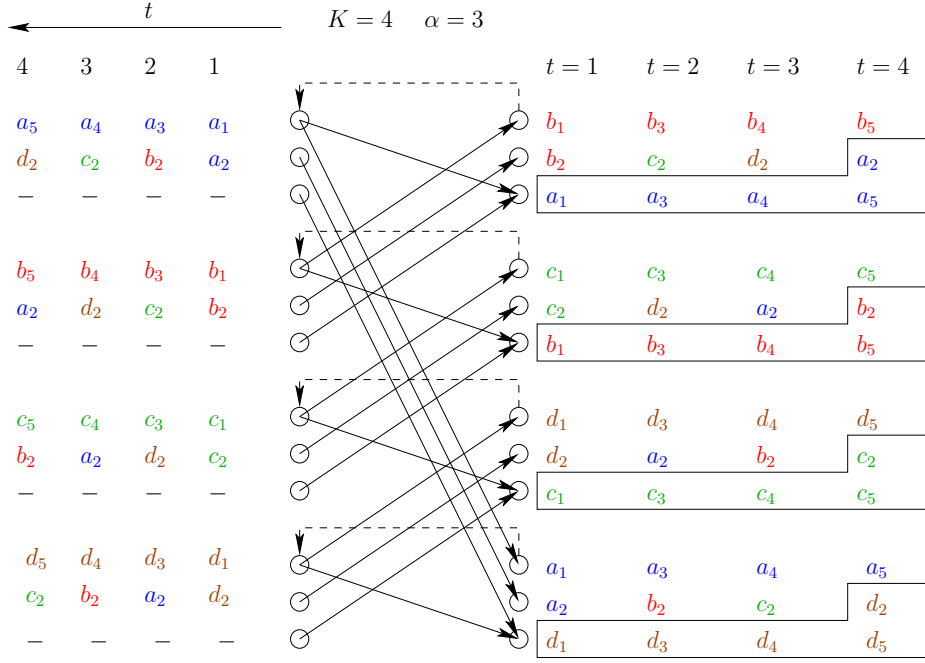


Figure 5: Feedback Coding scheme for $\alpha = 3$, $K = 4$.

To substantiate this claim, we note that when $\alpha \geq 2$ (corresponding to $m \geq 2n$), a normalized per-user rate of 1 can always be achieved without feedback by remaining silent on the lower most $(m - n)$ levels and sending fresh information in the top-most n levels. However, with feedback, each user can send additional information in the middle $(m - 2n)$ levels in the first channel use. This additional information can eventually reach the intended decoder via the delayed feedback path in K channel uses. For instance, in Figure 5, the bit a_2 is eventually received at decoder 1 in the last channel use. Therefore feedback can boost the per-user rate from Kn bits to $Kn + (m - 2n)$ bits in K uses of the channel. Therefore, the rate gain obtained via feedback is $(m - 2n)/K$ which decreases as K increases.

5 Upper bounds on the Feedback Sum-Capacity

In this section, we present two types of upper bounds on the sum-capacity of the K -user LD-CZIC. The type-I upper bound allows us to show that the normalized symmetric feedback capacity for the K -user LD-CZIC is always upper bounded by the symmetric feedback capacity of the 2-user system. The type-II upper bound is in fact a set of $K!$ genie-aided upper bounds, in which each upper bound corresponds to a permutation of K users. These type-II upper bounds are in fact valid for the general K -user interference channel with noiseless channel output feedback, i.e., they are not specifically derived for the cyclic interference channel.

We present the type-I upper bound in the following theorem:

Theorem 3 *The normalized symmetric feedback capacity of the K -user LD-CZIC satisfies*

$$\mathcal{C}_{\text{sym,LD}}^{\text{FB}}(\alpha, K) \leq \max\left(1 - \frac{\alpha}{2}, \frac{\alpha}{2}\right). \quad (27)$$

The proof of Theorem 3 is given in the appendix. The main idea behind this upper bound is to show that

$$R_j + R_{(j+1)} \leq \max(2n - m, m), \quad (28)$$

for $j = 1, \dots, K$. By adding all such K upper bounds and normalizing by $2nK$, we obtain the desired bound stated in Theorem 3. Theorem 3 along with (21) leads to the conclusion that feedback does not increase the symmetric capacity of the K -user LD-CZIC in the regime $\alpha \in [2/3, 2]$. We next present the type-II upper bound:

Theorem 4 *Fix a permutation order $\pi = \{\pi_1, \dots, \pi_K\}$, then the feedback sum-capacity of the general K -user interference channel is upper bounded as follows:*

$$\begin{aligned} & \mathcal{C}_{\text{sum}}^{\text{FB}}(K) \\ & \leq \max_{p(x_1, \dots, x_K)} \sum_{k=1}^K H(Y_{\pi_k} | X_{\pi_1}, Y_{\pi_1}, \dots, X_{\pi_{k-1}}, Y_{\pi_{k-1}}) - H(Y_1, \dots, Y_K | X_1, \dots, X_K). \end{aligned}$$

To illustrate by an example, consider the case when $K = 3$, for which Theorem 4 yields 6 upper bounds on the feedback sum capacity:

$$\begin{aligned} & \max_{p(x_1, x_2, x_3)} \left[H(Y_1) + H(Y_2 | X_1, Y_1) + H(Y_3 | X_1, X_2, Y_1, Y_2) - H(Y_1, Y_2, Y_3 | X_1, X_2, X_3) \right] \\ & \max_{p(x_1, x_2, x_3)} \left[H(Y_1) + H(Y_3 | X_1, Y_1) + H(Y_2 | X_1, X_3, Y_1, Y_3) - H(Y_1, Y_2, Y_3 | X_1, X_2, X_3) \right] \\ & \max_{p(x_1, x_2, x_3)} \left[H(Y_2) + H(Y_1 | X_2, Y_2) + H(Y_3 | X_1, X_2, Y_1, Y_2) - H(Y_1, Y_2, Y_3 | X_1, X_2, X_3) \right] \\ & \max_{p(x_1, x_2, x_3)} \left[H(Y_2) + H(Y_3 | X_2, Y_2) + H(Y_1 | X_2, X_3, Y_2, Y_3) - H(Y_1, Y_2, Y_3 | X_1, X_2, X_3) \right] \\ & \max_{p(x_1, x_2, x_3)} \left[H(Y_3) + H(Y_1 | X_3, Y_3) + H(Y_2 | X_1, X_3, Y_1, Y_3) - H(Y_1, Y_2, Y_3 | X_1, X_2, X_3) \right] \\ & \max_{p(x_1, x_2, x_3)} \left[H(Y_3) + H(Y_2 | X_3, Y_3) + H(Y_1 | X_2, X_3, Y_2, Y_3) - H(Y_1, Y_2, Y_3 | X_1, X_2, X_3) \right]. \end{aligned}$$

For an arbitrary K , Theorem 4 gives a total of $K!$ upper bounds. Optimization of these bounds for the general K user case and asymmetric channel gains is prohibitively complex. For the scope of this paper, we are interested in the case of CZIC with symmetric channel parameters. Depending on the range of the interference parameter α , we carefully select one of the type-II bounds and evaluate it to obtain the desired converse result as stated in Theorem 1.

5.1 Very Weak and Weak interference regimes: $0 \leq \alpha \leq 2/3$

In this regime, we select the type-II upper bound corresponding to the identical permutation order:

$$\pi = (1, 2, \dots, K). \quad (29)$$

Theorem 4 yields the following upper bound on the sum-capacity:

$$\mathcal{C}_{\text{sum,LD}}^{\text{FB}}(K) \leq \max_{p(x_1, \dots, x_K)} \sum_{k=1}^K H(Y_k | X_1, Y_1, \dots, X_{k-1}, Y_{k-1}) - H(Y_1, \dots, Y_K | X_1, \dots, X_K) \quad (30)$$

$$= \max_{p(x_1, \dots, x_K)} \sum_{k=1}^K H(Y_k | X_1, Y_1, \dots, X_{k-1}, Y_{k-1}) \quad (31)$$

$$= \max_{p(x_1, \dots, x_K)} H(Y_1) + H(Y_2 | X_1, Y_1) + \dots + H(Y_K | X_1, Y_1, \dots, X_{K-1}, Y_{K-1}) \quad (32)$$

$$\leq n + \max_{p(x_1, \dots, x_K)} \sum_{k=2}^{K-1} H(Y_k | X_{k-1}, Y_{k-1}) + \max_{p(x_1, \dots, x_K)} H(Y_K | X_1, Y_1, X_{K-1}, Y_{K-1}), \quad (33)$$

where (31) follows from the fact that (Y_1, \dots, Y_K) are all deterministic functions of (X_1, \dots, X_K) , and (33) follows from the fact that $H(Y_1) \leq \max(m, n) = n$.

To further upper bound (33), we first recall the notation used for $n \geq m$ in (5):

U_k : top-most $(n - m)$ bits of X_k

V_k : top-most m bits of X_k

L_k : lower-most m bits of X_k .

For any $2 \leq k \leq (K - 1)$, we have the following sequence of inequalities:

$$H(Y_k | X_{k-1}, Y_{k-1}) = H(Y_k | X_{k-1}, Y_{k-1}, V_k) \quad (34)$$

$$= H(U_k, L_k \oplus V_{(k+1)} | X_{k-1}, Y_{k-1}, V_k) \quad (35)$$

$$\leq H(U_k | V_k) + H(L_k \oplus V_{k+1}) \quad (36)$$

$$\leq \max(0, n - 2m) + m, \quad (37)$$

where (34) is due to the fact that V_k can be determined from (X_{k-1}, Y_{k-1}) .

Finally we upper bound the last term in (33) as follows:

$$H(Y_K|X_1, Y_1, X_{K-1}, Y_{K-1}) = H(Y_K|V_1, X_1, Y_1, X_{K-1}, Y_{K-1}, V_K) \quad (38)$$

$$= H(U_K, L_K \oplus V_1|V_1, V_K, X_1, Y_1, X_{K-1}, Y_{K-1}) \quad (39)$$

$$\leq H(U_K, L_K|V_K) \quad (40)$$

$$= H(X_K|V_K) \quad (41)$$

$$\leq (n - m). \quad (42)$$

Using (37) and (42), we can further upper bound (33) to obtain

$$\mathcal{C}_{\text{sum,LD}}^{\text{FB}}(K) \leq n + (K - 2) \left[\max(0, n - 2m) + m \right] + (n - m). \quad (43)$$

Therefore, the normalized symmetric feedback capacity is upper bounded as follows:

$$\mathcal{C}_{\text{sym,LD}}^{\text{FB}}(\alpha, K) \leq \max(\alpha, 1 - \alpha) + \frac{\min(\alpha, 2 - 3\alpha)}{K}, \quad (44)$$

which can also be written as

$$\mathcal{C}_{\text{sym,LD}}^{\text{FB}}(\alpha, K) \leq \begin{cases} (1 - \alpha) + \frac{\alpha}{K}, & 0 \leq \alpha \leq 1/2 \\ \alpha + \frac{(2-3\alpha)}{K}, & 1/2 \leq \alpha \leq 2/3. \end{cases} \quad (45)$$

Note that the upper bound alone shows that in the limit $K \rightarrow \infty$ the upper bound converges to the no-feedback symmetric capacity. This implies that in the limit of large K , the feedback gain vanishes.

5.2 Very strong interference: $\alpha \geq 2$

In this regime, we select the type-II upper bound corresponding to the following permutation order:

$$\pi = (1, K, K - 1, K - 2, \dots, 3, 2). \quad (46)$$

Theorem 4 yields the following upper bound on the sum-capacity:

$$\begin{aligned} \mathcal{C}_{\text{sum,LD}}^{\text{FB}}(K) &\leq \max_{p(x_1, \dots, x_K)} \left[H(Y_1) + H(Y_K|X_1, Y_1) + H(Y_{K-1}|X_1, X_K, Y_1, Y_K) + \dots \right. \\ &\quad \left. + H(Y_2|X_1, X_3, \dots, X_K, Y_1, Y_3, \dots, Y_K) \right] \end{aligned} \quad (47)$$

$$\begin{aligned} &\leq \max_{p(x_1, \dots, x_K)} \left[H(Y_1) + H(Y_K|X_1, Y_1) + \sum_{k=3}^{K-1} H(Y_k|X_{k+1}, Y_{k+1}) \right. \\ &\quad \left. + H(Y_2|X_1, Y_1, X_3, Y_3) \right]. \end{aligned} \quad (48)$$

To further upper bound (48), we recall the notation used for $n < m$ in (7):

U_k : top-most $(m - n)$ bits of X_k

V_k : top-most n bits of X_k

L_k : lower-most n bits of X_k .

We now upper bound the terms in (48) as follows. We first have the trivial upper bound $H(Y_1) \leq \max(m, n) = m$. We then bound the second term in (48) as follows:

$$H(Y_K|X_1, Y_1) = H(U_1, L_1 \oplus V_K|X_1, Y_1) \quad (49)$$

$$= H(L_1 \oplus V_K|X_1, Y_1) \quad (50)$$

$$\leq n. \quad (51)$$

Next, for any $3 \leq k \leq (K - 1)$, we have

$$H(Y_k|X_{k+1}, Y_{k+1}) = H(U_{k+1}, L_{k+1} \oplus V_k|X_{k+1}, Y_{k+1}) \quad (52)$$

$$= H(L_{k+1} \oplus V_k|X_{k+1}, Y_{k+1}) \quad (53)$$

$$\leq n, \quad (54)$$

which implies that

$$\sum_{k=3}^{K-1} H(Y_k|X_{k+1}, Y_{k+1}) \leq (K - 3)n. \quad (55)$$

Finally, we have

$$H(Y_2|X_1, Y_1, X_3, Y_3) = H(U_3, L_3 \oplus V_2|X_1, Y_1, X_3, Y_3) \quad (56)$$

$$= H(V_2|X_1, Y_1, X_3, Y_3) \quad (57)$$

$$= H(V_2|U_2, X_1, Y_2, X_3, Y_3) \quad (58)$$

$$= 0, \quad (59)$$

where (59) follows from the fact that $\alpha \geq 2$ corresponds to the case in which $m - n \geq n$ and therefore V_2 is completely determined by U_2 .

Using (51), (55) and (59), we have the following upper bound from (48):

$$\mathcal{C}_{\text{sum,LD}}^{\text{FB}}(K) \leq H(Y_1) + H(Y_K|X_1, Y_1) + \sum_{k=3}^{K-1} H(Y_k|X_{k+1}, Y_{k+1}) + H(Y_2|X_1, Y_1, X_3, Y_3) \quad (60)$$

$$\leq m + (K - 2)n. \quad (61)$$

Normalizing this upper bound by nK , we obtain

$$\mathcal{C}_{\text{sym,LD}}^{\text{FB}}(\alpha, K) = \frac{\mathcal{C}_{\text{sum,LD}}^{\text{FB}}(K)}{nK} \quad (62)$$

$$\leq \frac{m + (K - 2)n}{nK} \quad (63)$$

$$= 1 + \frac{(\alpha - 2)}{K}, \quad (64)$$

which is the desired upper bound on the normalized symmetric feedback capacity.

6 Gaussian K -user CZIC with Feedback

In this section, we consider the K -user Gaussian CZIC with feedback. The signal transmitted by user k is denoted by X_k . We impose an average unit power constraint at each user; that is $\mathbb{E}[X_k^2] \leq 1$. The signal observed at receiver k is obtained by

$$Y_k = \sqrt{\text{SNR}}X_k + \sqrt{\text{INR}}X_{k+1} + Z_k, \quad k = 1, 2, \dots, K, \quad (65)$$

where we define $X_{K+1} = X_1$ for consistency.

In the following, we study five different regimes depending on the parameter α (again, defined for this model as $\alpha = \log(\text{INR})/\log(\text{SNR})$), and propose upper bounds and feedback coding schemes for each one. We analyze the performance of the proposed schemes, and derive a symmetric achievable rate for them. In the rest of this section, we use bold symbols to denote blocks of length T , e.g.,

$$\mathbf{x}_k[j] = (X_k((j-1)T+1), (X_k((j-1)T+2), \dots, (X_k(jT))).$$

The encoding schemes that we propose for each regime involve message splitting and rate-power allocation to the resulting sub-messages. Whenever we use **SNR** or **INR** to determine a power allocation, we inherently assume that $\text{SNR} \geq 1$ and $\text{INR} \geq 1$, e.g., we allocate $\sqrt{\frac{\text{SNR}-1}{\text{SNR}}}$ of the power to some part of the message, which is meaningful only if **SNR** is larger than 1. However, if this assumption does not hold, we can still use the same coding schemes with $\max(1, \text{SNR})$ and $\max(1, \text{INR})$ instead of **SNR** and **INR** respectively, which leads to similar results. However, we exclude such cases for the sake of brevity.

We state our upper bounds in terms of following expressions:

$$A \triangleq \frac{1}{2} \log \left(1 + \text{SNR} + \text{INR} + 2\sqrt{\text{SNR} \cdot \text{INR}} \right) \quad (66)$$

$$B \triangleq \frac{1}{2} \log \left(1 + \text{SNR} + 2\text{INR} + \text{INR}^2 + 2\sqrt{\text{SNR} \cdot \text{INR}} \right) \quad (67)$$

$$C \triangleq \frac{1}{2} \log (1 + \text{SNR} + \text{INR}) \quad (68)$$

$$D \triangleq \frac{1}{2} \log (1 + \text{SNR}) \quad (69)$$

$$E \triangleq \frac{1}{2} \log (1 + \text{INR}). \quad (70)$$

6.1 Very Weak Interference: $0 \leq \alpha \leq 1/2$

6.1.1 Coding Scheme

The encoding scheme we propose here takes K blocks, each of length T . We assume each user k has a total of $(2K + 1)$ messages, namely

$$\left(M_k^{(h)}, M_k^{(1,m)}, M_k^{(2,m)}, \dots, M_k^{(K,m)}, M_k^{(1,l)}, M_k^{(2,l)}, M_k^{(K,l)} \right),$$

that it wishes to send to its respective receiver over K transmission blocks. Moreover, we set the size of these message sets with rates given by

$$\begin{aligned} \frac{\log |\mathcal{M}_k^{(h)}|}{n} &= R_1 = \frac{1}{2} \log \left(\frac{\text{INR} + 1}{3} \right), \quad k = 1, \dots, K, \\ \frac{\log |\mathcal{M}_k^{(j,m)}|}{n} &= R_2 = \frac{1}{2} \log \left(\frac{\frac{\text{SNR}}{\text{INR}} + 1}{\text{INR} + 1} \right), \quad k = 1, \dots, K, \quad j = 1, \dots, K, \\ \frac{\log |\mathcal{M}_k^{(j,l)}|}{n} &= R_3 = \frac{1}{2} \log \left(\frac{\text{INR} + 1}{2} \right), \quad k = 1, \dots, K, \quad j = 1, \dots, K. \end{aligned} \quad (71)$$

The messages are encoded using individual Gaussian codebooks with unit average power, to obtain

$$\left(\mathbf{s}_k^{(h)}, \mathbf{s}_k^{(1,m)}, \dots, \mathbf{s}_k^{(K,m)}, \mathbf{s}_k^{(1,l)}, \mathbf{s}_k^{(K,l)} \right).$$

The signal transmitted by user k in block k is formed as

$$\mathbf{x}_k[j] = \sqrt{\frac{\text{INR} - 1}{\text{INR}}} \mathbf{x}_{k,h}[j] + \sqrt{\frac{\text{SNR} - \text{INR}^2}{\text{SNR} \cdot \text{INR}}} \mathbf{x}_{k,m}[j] + \sqrt{\frac{\text{INR}}{\text{SNR}}} \mathbf{x}_{k,l}[j],$$

where $\mathbf{x}_{k,m}[j] = \mathbf{s}_k^{(j,m)}$ and $\mathbf{x}_{k,l} = \mathbf{s}_k^{(j,l)}$. In the first block, we have $\mathbf{x}_{k,h}[1] = \mathbf{s}_k^{(h)}$. For subsequent blocks, the high power part of the signal consists of the high power codeword of the neighboring transmitter sent over the last block, that is $\mathbf{x}_{k,h}[j] = \mathbf{x}_{k+1,h}[j-1]$. Of

course, this is only possible if transmitter k can decode $\mathbf{x}_{k+1,h}[j-1]$ from $\mathbf{y}_k[j-1]$ received over the feedback link.

Decoding the feedback signal at encoder Upon receiving $\mathbf{y}_k[j-1]$, the decoder removes its own signal to obtain

$$\begin{aligned} & \mathbf{y}_k[j-1] - \sqrt{\text{SNR}}\mathbf{x}_k[j-1] \\ &= \sqrt{\text{INR}}\mathbf{x}_{k+1}[j-1] + \mathbf{z}_k[j-1] \\ &= \sqrt{\text{INR}-1}\mathbf{x}_{k+1,h}[j-1] + \sqrt{\frac{\text{SNR}-\text{INR}^2}{\text{SNR}}}\mathbf{x}_{k+1,m}[j-1] + \sqrt{\frac{\text{INR}^2}{\text{SNR}}}\mathbf{x}_{k+1,l}[j-1] + \mathbf{z}_k[j-1]. \end{aligned} \quad (72)$$

It then decodes $\mathbf{x}_{k+1,h}[j-1]$, up to rate

$$R_1 \leq \frac{1}{2} \log \left(\frac{\text{INR}+1}{2} \right). \quad (73)$$

Decoding process at the receiver At the receiver node k , it has to decode $\mathbf{x}_{k,h}[j]$ and $\mathbf{x}_{k,m}[j]$. This is done using a sequential decode-and-remove scheme, which can support rates that satisfy

$$R_1 \leq \frac{1}{2} \log \left(\frac{\text{SNR} + \text{INR} + 1}{\frac{\text{SNR}}{\text{INR}} + \text{INR} + 1} \right) \quad (74)$$

$$R_2 \leq \frac{1}{2} \log \left(\frac{\frac{\text{SNR}}{\text{INR}} + \text{INR} + 1}{2\text{INR} + 1} \right). \quad (75)$$

The receiver also stores the remaining part of its received signal,

$$\tilde{\mathbf{y}}_k[j] = \sqrt{\text{INR}}\mathbf{x}_{k,l}[j] + \sqrt{\text{INR}-1}\mathbf{x}_{k+1,h}[j] + \sqrt{\frac{\text{SNR}-\text{INR}^2}{\text{SNR}}}\mathbf{x}_{k+1,m}[j] + \sqrt{\frac{\text{INR}^2}{\text{SNR}}}\mathbf{x}_{k+1,l}[j] + \mathbf{z}_k[j], \quad (76)$$

for further processing. In the next block, upon decoding $\mathbf{x}_{k,h}[j+1]$, it can use it to remove a part of the interference in $\tilde{\mathbf{y}}_k[j]$. Recall that $\mathbf{x}_{k,h}[j+1] = \mathbf{x}_{k+1,h}[j]$. Therefore, by removing $\mathbf{x}_{k+1,h}[j]$ from $\tilde{\mathbf{y}}_k[j]$, it can decode $\mathbf{x}_{k,l}[j]$, provided that its rate satisfies

$$R_3 \leq \frac{1}{2} \log \left(\frac{\text{INR}+2}{2} \right). \quad (77)$$

The total achievable rate would be

$$R_{\text{sym}} = \frac{R_1 + KR_2 + KR_3}{K} \quad (78)$$

$$= \frac{1}{2} \log \left(\frac{\text{SNR}}{\text{INR}} + 1 \right) + \frac{1}{2K} \log(1 + \text{INR}) - \frac{K + \log 3}{2K} \quad (79)$$

$$\geq (D - E) + \frac{E}{K} - \frac{K + \log 3}{2K}. \quad (80)$$

6.1.2 Upper Bound

In this regime, we use the following upper bound from Theorem 4 on the feedback sum-capacity:

$$\begin{aligned} \mathcal{C}_{\text{sum,G}}^{\text{FB}}(K) \leq \max_{p(x_1, \dots, x_K)} & \left[h(Y_1) + h(Y_2|X_1, Y_1) + \dots + h(Y_K|X_1, Y_1, \dots, X_{K-1}, Y_{K-1}) \right. \\ & \left. - h(Y_1, \dots, Y_K|X_1, \dots, X_K) \right]. \end{aligned} \quad (81)$$

We first note the following:

$$h(Y_1) \leq \frac{1}{2} \log \left(1 + \text{SNR} + \text{INR} + 2\sqrt{\text{SNR} \cdot \text{INR}} \right) + c \quad (82)$$

$$= A + c, \quad (83)$$

where $c = 1/2 \log(2\pi e)$.

For $2 \leq k \leq (K - 1)$, we have

$$h(Y_k|X_1, Y_1, \dots, X_{k-1}, Y_{k-1}) \leq h(Y_k|X_{k-1}, Y_{k-1}) \quad (84)$$

$$= h(\sqrt{\text{SNR}}X_k + \sqrt{\text{INR}}X_{k+1} + Z_k | \sqrt{\text{INR}}X_k + Z_{k-1}, X_{k-1}) \quad (85)$$

$$\leq h(\sqrt{\text{SNR}}X_k + \sqrt{\text{INR}}X_{k+1} + Z_k | \sqrt{\text{INR}}X_k + Z_{k-1}) \quad (86)$$

$$\leq \frac{1}{2} \log \left(\frac{1 + \text{SNR} + 2\text{INR} + \text{INR}^2 + 2\sqrt{\text{SNR} \cdot \text{INR}}}{1 + \text{INR}} \right) + c \quad (87)$$

$$= B - E + c. \quad (88)$$

Similarly, we have

$$h(Y_K|X_1, Y_1, \dots, X_{K-1}, Y_{K-1}) \leq h(Y_K|X_1, Y_1, X_{K-1}, Y_{K-1}) \quad (89)$$

$$\leq h(\sqrt{\text{SNR}}X_K + Z_K | \sqrt{\text{INR}}X_K + Z_{K-1}) \quad (90)$$

$$\leq \frac{1}{2} \log \left(\frac{1 + \text{SNR} + \text{INR}}{1 + \text{INR}} \right) + c \quad (91)$$

$$= C - E + c. \quad (92)$$

Finally, we have

$$h(Y_1, \dots, Y_K | X_1, \dots, X_K) = h(Z_1, \dots, Z_K) \quad (93)$$

$$= Kc. \quad (94)$$

Hence, from (81), we have

$$\mathcal{C}_{\text{sum,G}}^{\text{FB}}(K) \leq A + (K-2)(B-E) + C - E \quad (95)$$

$$= K(B-E) + (A+C+E-2B), \quad (96)$$

which implies that

$$\mathcal{C}_{\text{sym,G}}^{\text{FB}}(K) \leq (B-E) + \frac{A+C+E-2B}{K} \quad (97)$$

$$= (B-E) + \frac{E}{K} + \frac{(A+C-2B)}{K} \quad (98)$$

$$\leq (B-E) + \frac{E}{K}. \quad (99)$$

where we have used the fact that

$$A \leq B \quad (100)$$

$$C \leq B, \quad (101)$$

which implies that $(A+C-2B) \leq 0$.

We also note that

$$2B = \log(1 + \text{SNR} + 2\text{INR} + \text{INR}^2 + 2\sqrt{\text{SNR} \cdot \text{INR}}) \quad (102)$$

$$\leq \log(1 + 6\text{SNR}) \quad (103)$$

$$\leq \log(6) + \log(1 + \text{SNR}) \quad (104)$$

$$= \log(6) + 2D. \quad (105)$$

Collecting all the bounds, we have

$$\mathcal{C}_{\text{sym,G}}^{\text{FB}}(K) \leq (B-E) + \frac{E}{K} \quad (106)$$

$$\leq (D-E) + \frac{E}{K} + \frac{\log(6)}{2}. \quad (107)$$

Hence, the symmetric feedback capacity satisfies

$$\left[(D-E) + \frac{E}{K} \right] - \frac{K + \log 3}{2K} \leq \mathcal{C}_{\text{sym,G}}^{\text{FB}}(K) \leq \left[(D-E) + \frac{E}{K} \right] + \frac{\log(6)}{2} \quad (108)$$

so that the gap is given as

$$\Delta = \frac{\log(6)}{2} + \frac{K + \log(3)}{2K} \quad (109)$$

$$< \frac{3}{2} + \frac{2 + K}{2K} \quad (110)$$

$$\leq \frac{5}{2}, \quad (111)$$

and the degrees of freedom are given as

$$\text{DoF}^{\text{FB}}(\alpha, K) = (1 - \alpha) + \frac{\alpha}{K}, \quad 0 \leq \alpha \leq 1/2. \quad (112)$$

6.2 Weak Interference: $1/2 \leq \alpha \leq 2/3$

6.2.1 Coding Scheme

In this regime we have $\text{SNR}^{1/2} \leq \text{INR} \leq \text{SNR}^{2/3}$. The encoding scheme for this regime takes benefit of the feedback link. We create a cycle of length K consisting of all the interfering and feedback links. A part of the message of each user is conveyed through this cycle.

The encoding scheme is performed over K block. Assume each user has $2K + 1$ messages, namely

$$\left(M_k^{(1,h)}, M_k^{(2,h)}, \dots, M_k^{(K,h)}, M_k^{(m)}, M_k^{(1,l)}, M_k^{(2,l)}, M_k^{(K,l)} \right),$$

The rates of the messages are given by

$$\begin{aligned} \frac{\log |\mathcal{M}_k^{(j,h)}|}{n} &= R_1 = \frac{1}{2} \log \left(\frac{\text{INR} + 1}{2 \frac{\text{SNR}}{\text{INR}} + 4} \right), \quad k = 1, \dots, K, \quad j = 1, \dots, K, \\ \frac{\log |\mathcal{M}_k^{(m)}|}{n} &= R_2 = \frac{1}{2} \log \left(\frac{(1 + \text{SNR})^2}{2(1 + \text{INR})^3} \right), \quad k = 1, \dots, K, \\ \frac{\log |\mathcal{M}_k^{(j,l)}|}{n} &= R_3 = \frac{1}{2} \log \left(\frac{\frac{\text{SNR}}{\text{INR}} + 2}{2} \right), \quad k = 1, \dots, K, \quad j = 1, \dots, K. \end{aligned} \quad (113)$$

The k th transmitter encodes its message using a individual Gaussian codebook with unit average power, to obtain

$$\left(\mathbf{s}_k^{(1,h)}, \dots, \mathbf{s}_k^{(K,h)}, \mathbf{s}_k^{(m)}, \mathbf{s}_k^{(1,l)}, \mathbf{s}_k^{(K,l)} \right).$$

The transmitting signal in block k is formed as

$$\mathbf{x}_k[j] = \sqrt{\frac{\text{INR}^2 - \text{SNR}}{\text{INR}^2}} \mathbf{x}_{k,h}[j] + \sqrt{\frac{\text{SNR} - \text{INR}}{\text{INR}^2}} \mathbf{x}_{k,m}[j] + \sqrt{\frac{1}{\text{INR}}} \mathbf{x}_{k,l}[j],$$

where $\mathbf{x}_{k,h}[j] = \mathbf{s}_k^{(j,h)}$ and $\mathbf{x}_{k,l}[j] = \mathbf{s}_k^{(j,l)}$. The moderate power codeword transmitted during

the first block is the codeword corresponding to $M_k^{(m)}$. However, in the next blocks, $\mathbf{x}_{k,m}$ would be the moderate power codeword of the neighbor sent during the past block. More precisely,

$$\mathbf{x}_{k,m}[j] = \mathbf{x}_{k+1,m}[j-1] = \mathbf{x}_{k+2,m}[j-2] = \cdots = \mathbf{x}_{k+j-1,m}[1] = \mathbf{s}_{k+j-1}^{(m)}.$$

Note that $\mathbf{x}_{k+1,m}[j-1]$ has to be decoded from the signal sent to the transmitter k over the feedback link at the end of block $(j-1)$.

Decoding the feedback signal at encoder The signal received at receiver k in block j is forwarded to its respective transmitter at the end of the block. The transmitter will use it for forming its transmitting signal in the next block. The transmitter k removes the part of the signal sent by it to obtain

$$\begin{aligned} \mathbf{y}_k[j] - \sqrt{\text{SNR}}\mathbf{x}_k[j] &= \sqrt{\text{INR}}\mathbf{x}_{k+1}[j] + \mathbf{z}_k[j] \\ &= \sqrt{\frac{\text{INR}^2 - \text{SNR}}{\text{INR}}}\mathbf{x}_{k+1,h}[j] + \sqrt{\frac{\text{SNR} - \text{INR}}{\text{INR}}}\mathbf{x}_{k+1,m}[j] + \mathbf{x}_{k+1,l}[j] + \mathbf{z}_k[j]. \end{aligned} \quad (114)$$

The transmitter needs the moderate power codeword for the next transmission. In order to decode $\mathbf{x}_{k+1,m}[j]$, it first decodes and removes the high power codeword, and then decodes the moderate power one. This can be done provided

$$R_1 \leq \frac{1}{2} \log \left(\frac{\text{INR} + 1}{\frac{\text{SNR}}{\text{INR}} + 1} \right) \quad (115)$$

$$R_2 \leq \frac{1}{2} \log \left(\frac{\frac{\text{SNR}}{\text{INR}} + 1}{2} \right). \quad (116)$$

It is easy to check that the rates in (113) satisfy both constraints.

Decoding process at the receiver The decoding procedure at decoder k is as follows. Upon receiving

$$\begin{aligned} \mathbf{y}_k[j] &= \sqrt{\frac{\text{SNR}}{\text{INR}^2}}(\text{INR}^2 - \text{SNR})\mathbf{x}_{k,h}[j] + \sqrt{\frac{\text{SNR}}{\text{INR}^2}}(\text{SNR} - \text{INR})\mathbf{x}_{k,m}[j] + \sqrt{\frac{\text{INR}^2 - \text{SNR}}{\text{INR}}}\mathbf{x}_{k+1,h}[j] \\ &\quad + \sqrt{\frac{\text{SNR}}{\text{INR}}}\mathbf{x}_{k,l}[j] + \sqrt{\frac{\text{SNR} - \text{INR}}{\text{INR}}}\mathbf{x}_{k+1,m}[j] + \mathbf{x}_{k+1,l}[j] + \mathbf{z}_k[k] \end{aligned} \quad (117)$$

it decodes $\mathbf{x}_{k,h}[j]$, $\mathbf{x}_{k,m}[j]$, and $\mathbf{x}_{k+1,h}$ sequentially; that is in each step it treats everything else as noise, decodes the codeword, and removes the corresponding part from the received

signal. This can be done as long as

$$R_1 \leq \frac{1}{2} \log \left(\frac{\text{SNR} + \text{INR} + 1}{\frac{\text{SNR}^2}{\text{INR}^2} + \text{INR} + 1} \right) \quad (118)$$

$$R_2 \leq \frac{1}{2} \log \left(\frac{\frac{\text{SNR}^2}{\text{INR}^2} + \text{INR} + 1}{\frac{\text{SNR}}{\text{INR}} + \text{INR} + 1} \right) \quad (119)$$

$$R_1 \leq \frac{1}{2} \log \left(\frac{\frac{\text{SNR}}{\text{INR}} + \text{INR} + 1}{2\frac{\text{SNR}}{\text{INR}} + 1} \right), \quad (120)$$

which are all satisfied with the rates in (113). The remaining part of the signal would be

$$\tilde{\mathbf{y}}_k[j] = \sqrt{\frac{\text{SNR}}{\text{INR}}} \mathbf{x}_{k,l}[j] + \sqrt{\frac{\text{SNR} - \text{INR}}{\text{INR}}} \mathbf{x}_{k+1,m}[j] + \mathbf{x}_{k+1l,m}[j] + \mathbf{z}_k[k], \quad (121)$$

which will be stored for further processing and for decoding $\mathbf{x}_{k,l}[j]$ later. In the next block, once $\mathbf{x}_{k,m}[j+1]$ is decoded, the decoder again recalls $\tilde{\mathbf{y}}_k[j]$ and subtracts from it the part corresponding to $\mathbf{x}_{k,m}[j+1] = \mathbf{x}_{k+1,m}[j]$. Therefore, it obtains

$$\tilde{\mathbf{y}}_k[j] - \sqrt{\frac{\text{SNR} - \text{INR}}{\text{INR}}} \mathbf{x}_{k+1,m}[j] = \sqrt{\frac{\text{SNR}}{\text{INR}}} \mathbf{x}_{k,l}[j] + \mathbf{x}_{k+1,l}[j] + \mathbf{z}_k[k], \quad (122)$$

from which $\mathbf{x}_{k,l}[j]$ can be decoded as long as

$$R_3 \leq \frac{1}{2} \log \left(\frac{\frac{\text{SNR}}{\text{INR}} + 2}{2} \right), \quad (123)$$

which clearly holds with R_3 in (113). Therefore, in each block, the receiver can decode the low power codeword of the previous block after removing the moderate power interfering signal. However, this process does not have to continue for ever, since in the K -th block, the moderate power interfering signal at receiver k would be

$$\mathbf{x}_{k+1,m}[K] = \mathbf{x}_{k+2,m}[K-1] = \cdots = \mathbf{x}_{k+K,m}[1] = \mathbf{x}_{k,m}[1],$$

which was already decoded in the first block.

In summary, the total rate can be achieved per user per block would be

$$R_{\text{sym}} = \frac{KR_1 + R_2 + KR_3}{K} \quad (124)$$

$$= \frac{1}{2} \log(\text{INR} + 1) - 1 + \frac{1}{K} R_2 \quad (125)$$

$$= E + \frac{1}{2K} \log\left(\frac{(1 + \text{SNR})^2}{2(1 + \text{INR})^3}\right) - 1 \quad (126)$$

$$= E + \frac{(2D - 3E)}{K} - 1 - \frac{1}{2K}. \quad (127)$$

6.2.2 Upper Bound

In this regime, we use the same upper bound as in the case of very weak interference regime. In particular, from (96), we have

$$\mathcal{C}_{\text{sym,G}}^{\text{FB}}(K) \leq (B - E) + \frac{(A + C + E - 2B)}{K} \quad (128)$$

$$\leq 1 + E + \frac{(A + C + E - 4E)}{K} \quad (129)$$

$$\leq 1 + E + \frac{(1 + D + 1/2 + D + E - 4E)}{K} \quad (130)$$

$$= E + \frac{(2D - 3E)}{K} + 1 + \frac{3}{2K}. \quad (131)$$

Here, in (130), we have used the fact that for the weak interference regime, we have $2E \leq B \leq 1 + 2E$, $A \leq 1 + D$, and $C \leq 1/2 + D$. Therefore, we have

$$\left\lceil E + \frac{(2D - 3E)}{K} \right\rceil - 1 - \frac{1}{2K} \leq \mathcal{C}_{\text{sym,G}}^{\text{FB}}(K) \leq \left\lceil E + \frac{(2D - 3E)}{K} \right\rceil + 1 + \frac{3}{2K}, \quad (132)$$

which implies that the gap is bounded as follows:

$$\Delta \leq 2 + \frac{2}{K} \quad (133)$$

which is at most 3 bits per user-pair and we have

$$\mathbf{DoF}^{\text{FB}}(\alpha, K) = \alpha + \frac{(2 - 3\alpha)}{K}, \quad 1/2 \leq \alpha \leq 2/3. \quad (134)$$

6.3 Moderate Interference: $2/3 \leq \alpha \leq 1$

6.3.1 Coding Scheme

In this regime we use the private and common message for the encoding scheme. Assume each transmitter has two messages, namely the high power (common) message $M_k^{(h)}$, and the low power (private) message $M_k^{(l)}$. The following transmission scheme aims to convey

the common message $M_k^{(h)}$ to both receivers k and $k+1$. However, the private message $M_k^{(l)}$ can be decoded only by the respective receiver.

We assume that the high power messages of all users have the same rate. Similarly, the rate of the low power messages for all users are the same, that is

$$\begin{aligned} R_1 &= \frac{\log |\mathcal{M}_k^{(h)}|}{n}, & k = 1, \dots, K, \\ R_2 &= \frac{\log |\mathcal{M}_k^{(l)}|}{n}, & k = 1, \dots, K. \end{aligned} \quad (135)$$

The encoder first maps its messages to Gaussian codewords with unit average power, $\mathbf{x}_{k,h}$ and $\mathbf{x}_{k,l}$, and sends

$$\mathbf{x}_k = \sqrt{\frac{\text{INR} - 1}{\text{INR}}} \mathbf{x}_{k,h} + \sqrt{\frac{1}{\text{INR}}} \mathbf{x}_{k,l}.$$

The receiver node k , upon receiving \mathbf{y}_k , with

$$\mathbf{y}_k[j] = \sqrt{\text{SNR}} \mathbf{x}_k[j] + \sqrt{\text{INR}} \mathbf{x}_{k+1}[j] + \mathbf{z}_k[j] \quad (136)$$

$$= \sqrt{\frac{\text{SNR}}{\text{INR}}} (\text{INR} - 1) \mathbf{x}_{k,h}[j] + \sqrt{\text{INR} - 1} \mathbf{x}_{k+1,h}[j] + \sqrt{\frac{\text{SNR}}{\text{INR}}} \mathbf{x}_{k,l}[j] + \mathbf{x}_{k+1,l}[j] + \mathbf{z}_k[j], \quad (137)$$

first jointly decodes the codewords $\mathbf{x}_{k,h}^{(h)}$ and $\mathbf{x}_{k+1,h}^{(h)}$ treating all the rest as noise. Here we deal with a multiple access channel, whose achievable rate is characterized by

$$R_1 \leq \frac{1}{2} \log \left(\frac{\text{SNR} + 2}{\frac{\text{SNR}}{\text{INR}} + 2} \right), \quad (138)$$

$$R_1 \leq \frac{1}{2} \log \left(\frac{\text{INR} + \frac{\text{SNR}}{\text{INR}} + 1}{\frac{\text{SNR}}{\text{INR}} + 2} \right), \quad (139)$$

$$2R_1 \leq \frac{1}{2} \log \left(\frac{\text{SNR} + \text{INR} + 1}{\frac{\text{SNR}}{\text{INR}} + 2} \right). \quad (140)$$

In particular, it is easy to show that

$$R_1 = \frac{1}{4} \log \left(\frac{\text{SNR} + \text{INR} + 1}{\frac{\text{SNR}}{\text{INR}} + 2} \right) \quad (141)$$

satisfies the above constraints. After decoding the high power codewords, and removing them from the received signal, receiver k decodes its own low power message by treating the

other private codeword as noise. This private message can be reliably decoded provided that

$$R_2 \leq \frac{1}{2} \log \left(\frac{\frac{\text{SNR}}{\text{INR}} + 2}{2} \right), \quad (142)$$

which yields in an achievable total rate of

$$R_{\text{sym}} = R_1 + R_2 \quad (143)$$

$$= \frac{1}{4} \log(\text{SNR} + \text{INR} + 1) + \frac{1}{4} \log \left(2 + \frac{\text{SNR}}{\text{INR}} \right) - \frac{1}{2} \quad (144)$$

$$\geq \frac{1}{2} \log(1 + \text{SNR}) - \frac{1}{4} \log(1 + \text{INR}) - \frac{1}{2} \quad (145)$$

$$= D - \frac{E}{2} - \frac{1}{2}. \quad (146)$$

6.3.2 Upper Bound

In this regime, we will develop a different upper bound that is analogous to the type-I upper bound obtained for the linear deterministic channel model. We have the following bound on the feedback sum capacity:

$$\mathcal{C}_{\text{sum,G}}^{\text{FB}}(K) \leq \frac{K}{2}(A + C - E). \quad (147)$$

The proof of (147) is given in the appendix. Hence, (147) implies that the symmetric feedback capacity is upper bounded as

$$\mathcal{C}_{\text{sym,G}}^{\text{FB}}(K) \leq \frac{A + C - E}{2} \quad (148)$$

$$= \frac{A + C}{2} - \frac{E}{2}. \quad (149)$$

Note that in this regime, we have

$$2A = \log(1 + \text{SNR} + \text{INR} + 2\sqrt{\text{SNR} \cdot \text{INR}}) \quad (150)$$

$$\leq \log(1 + 4\text{SNR}) \quad (151)$$

$$\leq \log(4) + \log(1 + \text{SNR}) \quad (152)$$

$$= 2 + 2D, \quad (153)$$

and similarly,

$$2C = \log(1 + \text{SNR} + \text{INR}) \quad (154)$$

$$\leq \log(1 + 2\text{SNR}) \quad (155)$$

$$\leq \log(2) + \log(1 + \text{SNR}) \quad (156)$$

$$= 1 + 2D, \quad (157)$$

which implies that

$$\frac{A + C}{2} \leq D + \frac{3}{4}. \quad (158)$$

Hence, we have

$$\left[D - \frac{E}{2} \right] - \frac{1}{2} \leq \mathcal{C}_{\text{sym,G}}^{\text{FB}}(K) \leq \left[D - \frac{E}{2} \right] + \frac{3}{4}, \quad (159)$$

so that the gap between the upper and lower bounds is at most 5/4 bits and we have

$$\mathbf{DoF}^{\text{FB}}(\alpha, K) = 1 - \frac{\alpha}{2}, \quad 2/3 \leq \alpha \leq 1. \quad (160)$$

6.4 Strong Interference: $1 \leq \alpha \leq 2$

6.4.1 Coding Scheme

In this regime, we have $\text{SNR} \leq \text{INR} \leq \text{SNR}^2$. The encoding scheme for this interference regime is simple, and the desired degrees of freedom can be achieved in one block. Denote the message of user k by $M_k \in \mathcal{M}_k$, where all the message sets have the same size which results in a symmetric rate of R . Each user takes a random Gaussian codebook with rate R and unit average power. Then it randomly maps its message to \mathbf{x}_k and sends over the channel. The k -th receiver observes \mathbf{y}_k through a multiple access channel from the k -th and $(k+1)$ -th transmitters, in which it has to decode both messages. The achievable rate of the MAC is well-known as [11]

$$R \leq \frac{1}{2} \log(\text{SNR} + 1), \quad (161)$$

$$R \leq \frac{1}{2} \log(\text{INR} + 1), \quad (162)$$

$$2R \leq \frac{1}{2} \log(\text{INR} + \text{SNR} + 1). \quad (163)$$

Hence, it is clear that by choosing

$$R_{\text{sym}} = \frac{1}{4} \log(1 + \text{INR} + \text{SNR}) \quad (164)$$

$$= \frac{C}{2}, \quad (165)$$

all constraints are satisfied and a symmetric rate of R_{sym} is therefore achievable.

6.4.2 Upper bound

For this regime, we use the same upper bound developed in the previous section:

$$\mathcal{C}_{\text{sym,G}}^{\text{FB}}(K) \leq \frac{A + C - E}{2}. \quad (166)$$

Therefore, the symmetric feedback capacity satisfies

$$\frac{C}{2} \leq \mathcal{C}_{\text{sym,G}}^{\text{FB}}(K) \leq \frac{C}{2} + \frac{A - E}{2}, \quad (167)$$

and the gap between the bounds is

$$\Delta = \frac{A - E}{2} \quad (168)$$

$$= \frac{\log(1 + \text{SNR} + \text{INR} + 2\sqrt{\text{SNR} \cdot \text{INR}}) - \log(1 + \text{INR})}{4} \quad (169)$$

$$\leq \frac{\log(1 + 4\text{INR}) - \log(1 + \text{INR})}{4} \quad (170)$$

$$\leq \frac{\log(4) + \log(1 + \text{INR}) - \log(1 + \text{INR})}{4} \quad (171)$$

$$= \frac{1}{2}. \quad (172)$$

Moreover, from (167), it is straightforward to show that

$$\text{DoF}^{\text{FB}}(\alpha, K) = \frac{\alpha}{2}, \quad 1 \leq \alpha \leq 2. \quad (173)$$

6.5 Very-Strong Interference: $\alpha \geq 2$

6.5.1 Coding Scheme

The encoding scheme we propose here takes K blocks, each of length T . We assume each user k has a total of $K + 1$ messages, namely $(M_k^{(l)}, M_k^{(1,h)}, \dots, M_k^{(K,h)})$, that it wishes to send to its respective receiver over K transmission blocks. We assume that $M_k^{(l)} \in \mathcal{M}_k^{(l)}$ and $M_k^{(j,h)} \in \mathcal{M}_k^{(j,h)}$, where \mathcal{M} 's are the message sets. Moreover, we set the size of these message

sets so that

$$\begin{aligned}\frac{\log |\mathcal{M}_k^{(l)}|}{n} &= R_1 = \frac{1}{2} \log \left(\frac{\frac{\text{INR}}{\text{SNR}^2} + 1}{2} \right), & k = 1, \dots, K \\ \frac{\log |\mathcal{M}_k^{(j,h)}|}{n} &= R_2 = \frac{1}{2} \log \left(\frac{\text{SNR} + 1}{3} \right), & k = 1, \dots, K, j = 1, \dots, K.\end{aligned}\tag{174}$$

That is, all the first messages of all the users have the same rate. Furthermore, the rates of all the remaining messages are also identical. Each message is encoded by a capacity achieving Gaussian codebook with unit variance. Hence user k has got $K + 1$ Gaussian codewords, $\mathbf{s}_k^{(l)}, \mathbf{s}_k^{(1,h)}, \dots, \mathbf{s}_k^{(K,h)}$, each of length n .

The signal sent by transmitter k in block j is composed of two parts, the high power part $\mathbf{x}_{k,h}[j]$ and low power $\mathbf{x}_{k,l}[j]$:

$$\mathbf{x}_k[j] = \sqrt{\frac{\text{SNR} - 1}{\text{SNR}}} \mathbf{x}_{k,h}[j] + \sqrt{\frac{1}{\text{SNR}}} \mathbf{x}_{k,l}[j].\tag{175}$$

In all blocks, the high power part is the codeword corresponding to a fresh message. In the first block, since the nodes have not yet received any feedback, their low power codewords also describe a fresh message. However, for all blocks $j \geq 2$, the low level codeword used to form the transmitting signal is the low level codeword of their neighbor sent on the previous block. We will show that it can be decoded from the signal received over the feedback link at the end of the last block. More precisely,

$$\mathbf{x}_{k,h}[j] = \mathbf{s}_k^{(j,h)}, \quad k = 1, 2, \dots, K \quad j = 1, 2, \dots, K,\tag{176}$$

$$\mathbf{x}_{k,l}[1] = \mathbf{s}_k^{(l)}, \quad k = 1, 2, \dots, K,\tag{177}$$

$$\mathbf{x}_{k,l}[j] = \mathbf{x}_{k+1,l}[j-1], \quad k = 1, 2, \dots, K, \quad j = 2, 3, \dots, K.\tag{178}$$

Therefore we have the following recursive relationship between the low power codewords:

$$\mathbf{x}_{k,l}[j] = \mathbf{x}_{k+1,l}[j-1] = \mathbf{x}_{k+2,l}[j-2] = \dots = \mathbf{x}_{k+j-1,l}[1] = \mathbf{s}_{k+j-1}^{(l)},\tag{179}$$

where all the user and block indicators are modulo K , e.g., $\mathbf{x}_{k+j-1}[1] = \mathbf{x}_{(k+j-1) \bmod K}[1]$.

Decoding the feedback signal at encoder As stated above, in order to form the transmitting signal in block j , transmitter k uses the low power codeword sent by user $k + 1$ in block $j - 1$. We first show that this codeword can be decoded based on the signal it receives over the feedback link at the end of block $j - 1$.

Once $\mathbf{y}_k[j-1]$ is received, transmitter k first removes its own signal, $\mathbf{x}_k[j-1]$, to obtain

$$\begin{aligned}\mathbf{y}_k[j-1] - \sqrt{\text{SNR}}\mathbf{x}_k[j-1] &= \sqrt{\text{INR}}\mathbf{x}_{k+1}[j-1] + \mathbf{z}_k[j-1] \\ &= \sqrt{\frac{\text{INR}}{\text{SNR}}}(\text{SNR} - 1)\mathbf{x}_{k+1,h}[j-1] + \sqrt{\frac{\text{INR}}{\text{SNR}}}\mathbf{x}_{k+1,l}[j-1] + \mathbf{z}_k[j-1],\end{aligned}\tag{180}$$

$$\tag{181}$$

from which it has to decode both $\mathbf{x}_{k+1,h}[j-1]$ and $\mathbf{x}_{k+1,l}[j-1]$. It first decodes $\mathbf{x}_{k+1,h}[j-1]$, treating everything else as noise. Then, it removes $\mathbf{x}_{k+1,h}[j-1]$ from the signal and decodes $\mathbf{x}_{k+1,l}[j-1]$ in a similar manner. This is possible as long as

$$R_2 \leq \frac{1}{2} \log \left(\frac{\text{INR} + 1}{\frac{\text{INR}}{\text{SNR}} + 1} \right) \tag{182}$$

$$R_1 \leq \frac{1}{2} \log \left(\frac{\text{INR}}{\text{SNR}} + 1 \right), \tag{183}$$

which are clearly satisfied by the rates chosen in (174). Therefore, the transmitter k has access to $\mathbf{x}_{k+1,l}[j-1]$, which will be used as its low power codeword for block j .

Decoding process at the receiver The signal sent by user k over the j -th block is given by

$$\mathbf{x}_k[j] = \sqrt{\frac{\text{SNR} - 1}{\text{SNR}}}\mathbf{x}_{k,h}[j] + \sqrt{\frac{1}{\text{SNR}}}\mathbf{x}_{k,l}[j], \tag{184}$$

which results in

$$\begin{aligned}\mathbf{y}_k[j] &= \sqrt{\text{SNR}}\mathbf{x}_k[j] + \sqrt{\text{INR}}\mathbf{x}_{k+1}[j] + \mathbf{z}_k[j] \\ &= \sqrt{\frac{\text{INR}}{\text{SNR}}}(\text{SNR} - 1)\mathbf{x}_{k+1,h}[j] + \sqrt{\frac{\text{INR}}{\text{SNR}}}\mathbf{x}_{k+1,l}[j] + \sqrt{\text{SNR} - 1}\mathbf{x}_{k,h}[j] + \mathbf{x}_{k,l}[j] + \mathbf{z}_k[j].\end{aligned}\tag{185}$$

$$\tag{186}$$

At the end of the j -th block, user k sequentially decodes the codewords $\mathbf{x}_{k+1,h}[j]$, $\mathbf{x}_{k+1,l}[j]$, and $\mathbf{x}_{k,h}[j]$. At each step, it decodes the corresponding codewords, treating all the remaining parts as noise. Once one codeword is decoded, it removes it from its received signal, and

proceeds with the next codeword. This can be done provided that

$$R_2 \leq \frac{1}{2} \log \left(\frac{\text{INR} + \text{SNR} + 1}{\frac{\text{INR}}{\text{SNR}} + \text{SNR} + 1} \right), \quad (187)$$

$$R_1 \leq \frac{1}{2} \log \left(\frac{\frac{\text{INR}}{\text{SNR}} + \text{SNR} + 1}{\text{SNR} + 1} \right), \quad (188)$$

$$R_2 \leq \frac{1}{2} \log \left(\frac{\text{SNR} + 1}{2} \right). \quad (189)$$

It is easy to check that all constraints are satisfied by the choice of R_1 and R_2 in (174).

At the end of each block, each receiver can decode its respective high power codeword, as well as some high power and low power codewords from other users which it is not intended to decode. However, from (179), the low power codeword decoded at receiver k at the very last block would be

$$\mathbf{x}_{k+1,l}[K] = \mathbf{x}_{k+2,l}[K-1] = \dots = \mathbf{x}_{k+K,l}[1] \stackrel{(*)}{=} \mathbf{x}_{k,l}[1] = \mathbf{s}_k^{(l)},$$

where $(*)$ holds since $k+K = k \pmod K$. Therefore all the intended messages for receiver k , can be decoded using this scheme in K blocks. The total rate of communication would be

$$R_{\text{sym}} = \frac{R_1 + KR_2}{K} = \frac{1}{2} \log(\text{SNR} + 1) + \frac{1}{2K} \log \left(\frac{\text{INR}}{\text{SNR}^2} + 1 \right) - \frac{K \log 3 + 1}{2K} \quad (190)$$

$$= D + \frac{1}{2K} \log \left(\frac{\text{INR}}{\text{SNR}^2} + 1 \right) - \frac{K \log 3 + 1}{2K} \quad (191)$$

$$\geq D + \frac{1}{2K} \log \left(\frac{1 + \text{INR}}{(1 + \text{SNR})^2} \right) - \frac{K \log 3 + 1}{2K} \quad (192)$$

$$= D + \frac{(E - 2D)}{K} - \frac{(K \log 3 + 1)}{2K}. \quad (193)$$

6.5.2 Upper Bound

For this regime we use the following upper bound from Theorem 4, similar to the LD case:

$$\begin{aligned} \mathcal{C}_{\text{sum,G}}^{\text{FB}}(K) \leq \max_{p(x_1, \dots, x_K)} & \left[h(Y_K) + h(Y_{K-1}|X_K, Y_K) + \dots + h(Y_1|X_2, Y_2, \dots, X_K, Y_K) \right. \\ & \left. - h(Y_1, \dots, Y_K|X_1, \dots, X_K) \right]. \end{aligned} \quad (194)$$

We upper bound the first term in (194) as

$$h(Y_K) \leq \frac{1}{2} \log \left(1 + \text{SNR} + \text{INR} + 2\sqrt{\text{SNR} \cdot \text{INR}} \right) + c \quad (195)$$

$$= A + c, \quad (196)$$

where $c = (1/2) \log(2\pi e)$.

For any $2 \leq k \leq (K - 1)$, we bound

$$h(Y_k|X_{k+1}, Y_{k+1}, \dots, X_K, Y_K) \leq h(Y_k|X_{k+1}) \quad (197)$$

$$= h(\sqrt{\text{SNR}}X_k + Z_k|X_{k+1}) \quad (198)$$

$$\leq h(\sqrt{\text{SNR}}X_k + Z_k) \quad (199)$$

$$\leq \frac{1}{2} \log(1 + \text{SNR}) + c \quad (200)$$

$$= D + c. \quad (201)$$

Finally, we bound the penultimate term in (194) as follows:

$$h(Y_1|X_2, Y_2, \dots, X_K, Y_K) \leq h(Y_1|X_2, Y_2, X_K, Y_K) \quad (202)$$

$$= h(\sqrt{\text{SNR}}X_1 + Z_1|X_2, Y_2, X_K, Y_K) \quad (203)$$

$$\leq h(\sqrt{\text{SNR}}X_1 + Z_1|\sqrt{\text{INR}}X_1 + Z_K) \quad (204)$$

$$\leq \frac{1}{2} \log\left(\frac{1 + \text{SNR} + \text{INR}}{1 + \text{INR}}\right) + c \quad (205)$$

$$= C - E + c, \quad (206)$$

where in (204), we used the fact that $Y_K = \sqrt{\text{SNR}}X_K + \sqrt{\text{INR}}X_1 + Z_K$ and the fact that conditioning reduces differential entropy. Finally, we note that

$$h(Y_1, \dots, Y_K|X_1, \dots, X_K) = h(Z_1, \dots, Z_K|X_1, \dots, X_K) \quad (207)$$

$$= h(Z_1, \dots, Z_K) \quad (208)$$

$$= \sum_{k=1}^K h(Z_k) \quad (209)$$

$$= Kc. \quad (210)$$

Hence, the feedback sum capacity is upper bounded as follows

$$\mathcal{C}_{\text{sum,G}}^{\text{FB}}(K) \leq A + (K - 2)D + C - E, \quad (211)$$

which implies that the symmetric feedback capacity satisfies

$$\mathcal{C}_{\text{sym,G}}^{\text{FB}}(K) \leq D + \frac{(A + C - 2D - E)}{K}. \quad (212)$$

We now simplify this upper bound to compare it with the lower bound obtained in (193).

We note that

$$2A = \log(1 + \text{SNR} + \text{INR} + 2\sqrt{\text{SNR} \cdot \text{INR}}) \quad (213)$$

$$\leq \log(1 + 4\text{INR}) \quad (214)$$

$$\leq \log(4) + \log(1 + \text{INR}) \quad (215)$$

$$= 2 + 2E, \quad (216)$$

and

$$2C = \log(1 + \text{SNR} + \text{INR}) \quad (217)$$

$$\leq \log(1 + 2\text{INR}) \quad (218)$$

$$\leq \log(2) + \log(1 + \text{INR}) \quad (219)$$

$$= 1 + 2E, \quad (220)$$

which together imply that

$$A + C - 2D - E \leq (3/2) + 2E - 2D - E \quad (221)$$

$$= (E - 2D) + 3/2. \quad (222)$$

Hence, from (193) and (212), the symmetric feedback capacity satisfies

$$D + \frac{(E - 2D)}{K} - \frac{(K \log(3) + 1)}{2K} \leq C_{\text{sym,G}}^{\text{FB}}(K) \leq D + \frac{(E - 2D)}{K} + \frac{3}{2K} \quad (223)$$

which implies that the gap is given as

$$\Delta = \frac{4 + K \log(3)}{2K} \quad (224)$$

$$\leq \frac{2 + K}{K} \quad (225)$$

which is at most 2-bits. We note here that the gap of 2-bits can be reduced further to 1-bit by modifying the power allocation in our coding scheme. The resulting gap analysis is however complicated and is not pursued here.

Moreover, from (223), we have

$$\text{DoF}^{\text{FB}}(\alpha, K) = 1 + \frac{(\alpha - 2)}{K}, \quad \alpha \geq 2. \quad (226)$$

7 Conclusions

In this paper, we have considered the K -user cyclic Z-interference channel with noiseless feedback. The symmetric feedback capacity of the linear deterministic CZIC has been com-

pletely characterized for all interference regimes. Using insights from the linear model, the symmetric feedback capacity for the Gaussian CZIC has been characterized within a constant number of bits for all interference regimes. As a consequence of the constant bit gap result, the symmetric feedback degrees of freedom for the Gaussian CZIC has also been characterized. It has been shown that the capacity gain obtained via feedback decreases as the number of users increases. The resulting $\mathbf{DoF}^{\text{FB}}(\alpha, K)$ for $K > 2$ users is a skewed V -curve, as a function of the interference parameter α . Moreover as $K \rightarrow \infty$, the resulting skewed V -curve converges to the well known W -curve corresponding to the no-feedback \mathbf{DoF} .

As a part of future work, we plan to characterize the approximate feedback capacity region of the K -user Gaussian CZIC. We believe that new coding schemes and novel upper bounds would be required to achieve this goal.

8 Appendix

8.1 Proof of Theorem 3

We show that the normalized symmetric feedback capacity is upper bounded as follows:

$$\mathcal{C}_{\text{sym,LD}}^{\text{FB}}(\alpha, K) \leq \max\left(1 - \frac{\alpha}{2}, \frac{\alpha}{2}\right). \quad (227)$$

To prove (227), we first prove the following upper bound on the sum of the rates of users 1 and 2:

$$\begin{aligned} T(R_1 + R_2) &= H(W_1) + H(W_2) \end{aligned} \quad (228)$$

$$= H(W_1|W_3, \dots, W_K) + H(W_2|W_1, W_3, \dots, W_K) \quad (229)$$

$$\leq I(W_1; Y_1^T | W_3, \dots, W_K) + I(W_2; Y_2^T, Y_1^T | W_1, W_3, \dots, W_K) + \epsilon_T \quad (230)$$

$$= I(W_1; Y_1^T | W_3, \dots, W_K) + H(Y_2^T, Y_1^T | W_1, W_3, \dots, W_K) + \epsilon_T \quad (231)$$

$$= H(Y_1^T | W_3, \dots, W_K) + H(Y_2^T | Y_1^T, W_1, W_3, \dots, W_K) + \epsilon_T \quad (232)$$

$$\leq H(Y_1^T) + H(Y_2^T | Y_1^T, W_1, W_3, \dots, W_K) + \epsilon_T \quad (233)$$

$$\leq T \max(m, n) + H(Y_2^T | Y_1^T, W_1, W_3, \dots, W_K) + \epsilon_T \quad (234)$$

$$\leq T \max(m, n) + T(n - m)^+ + \epsilon_T, \quad (235)$$

where (229) follows from the fact that the messages (W_1, \dots, W_K) are all mutually independent, (230) follows from Fano's inequality [11], (231) follows from the deterministic nature of the channel model and (233) follows from the fact that conditioning reduces entropy.

Before proving (235) we first prove the following claim:

Claim 1 $(X_{1t}, X_{3t}, \dots, X_{Kt})$ is as deterministic function of $(Y_1^{t-1}, W_1, W_3, \dots, W_K)$.

Proof: First note that from (3), we have

$$X_{1t} = f_{1t}(W_1, Y_1^{t-1}), \quad (236)$$

and

$$X_{Kt} = f_{Kt}(W_K, Y_K^{t-1}) \quad (237)$$

$$= f_{Kt}(W_K, X_1^{t-1}, X_K^{t-1}), \quad (238)$$

which together imply that

$$(X_{1t}, X_{Kt}, X_1^{t-1}, X_K^{t-1}) = f(W_1, W_K, Y_1^{t-1}). \quad (239)$$

Repeating this argument for $k = K - 1, \dots, 3$, the proof of the claim is straightforward. ■

We now bound the second term in (234) as follows:

$$H(Y_2^T | Y_1^T, W_1, W_3, \dots, W_K) \leq \sum_{t=1}^T H(Y_{2t} | Y_{1t}, W_1, W_3, \dots, W_K, Y_1^{t-1}) \quad (240)$$

$$= \sum_{t=1}^T H(Y_{2t} | Y_{1t}, X_{1t}, W_1, X_{3t}, W_3, \dots, X_{Kt}, W_K, Y_1^{t-1}) \quad (241)$$

$$\leq \sum_{t=1}^T H(Y_{2t} | Y_{1t}, X_{1t}, X_{3t}) \quad (242)$$

$$\leq \sum_{t=1}^T H(X_{2t} | Y_{1t}, X_{1t}, X_{3t}) \quad (243)$$

$$\leq T(n - m)^+, \quad (244)$$

where (241) follows from Claim 1 and (244) follows from the fact that (X_{1t}, Y_{1t}) completely determine at least m levels of X_{2t} . This completes the proof of (235). Dividing (235) by nT and taking the limit $T \rightarrow \infty$, we have $\epsilon_T \rightarrow 0$, which yields

$$\frac{R_1 + R_2}{n} \leq \max\left(\frac{m}{n}, 1\right) + \left(1 - \frac{m}{n}\right)^+ \quad (245)$$

$$= \max(\alpha, 1) + (1 - \alpha)^+ \quad (246)$$

$$= \max(2 - \alpha, \alpha). \quad (247)$$

In a similar manner it can be shown that for any $1 \leq j \leq K$,

$$\frac{R_j + R_{(j+1) \bmod(K)}}{n} \leq \max(2 - \alpha, \alpha). \quad (248)$$

Adding all such K upper bounds, we obtain

$$\frac{2(R_1 + \dots + R_K)}{n} \leq K \max(2 - \alpha, \alpha), \quad (249)$$

and hence,

$$\mathcal{C}_{\text{sym,LD}}^{\text{FB}}(\alpha, K) \leq \max\left(1 - \frac{\alpha}{2}, \frac{\alpha}{2}\right). \quad (250)$$

This upper bound on the normalized symmetric feedback capacity is independent of K and is the same as the normalized symmetric capacity *without feedback* when $\alpha \in [2/3, 2]$. Hence, for this interference regime, feedback does not increase the symmetric capacity. Also note that the range of α in deriving these bounds is immaterial and hence from a symmetric feedback capacity point of view, the feedback capacity for $K = 2$ users always serves as an upper bound for any $K > 2$.

8.2 Proof of Theorem 4

For a permutation order $\pi = (\pi_1, \pi_2, \dots, \pi_K)$ for the K users, we have the following upper bound on the sum-rate:

$$T \left(\sum_{k=1}^K R_{\pi_k} \right) = \sum_{k=1}^K H(W_{\pi_k}) \quad (251)$$

$$= \sum_{k=1}^K H(W_{\pi_k} | W_{\pi_1}, \dots, W_{\pi_{k-1}}) \quad (252)$$

$$\leq \sum_{k=1}^K I(W_{\pi_k}; Y_{\pi_1}^T, \dots, Y_{\pi_k}^T | W_{\pi_1}, \dots, W_{\pi_{k-1}}) + \epsilon_T \quad (253)$$

$$= \sum_{k=1}^K \left[H(Y_{\pi_1}^T, \dots, Y_{\pi_k}^T | W_{\pi_1}, \dots, W_{\pi_{k-1}}) \right. \\ \left. - H(Y_{\pi_1}^T, \dots, Y_{\pi_k}^T | W_{\pi_1}, \dots, W_{\pi_{k-1}}, W_k) \right] + \epsilon_T \quad (254)$$

$$= \sum_{k=1}^K H(Y_{\pi_k}^T | Y_{\pi_1}^T, \dots, Y_{\pi_{k-1}}^T, W_{\pi_1}, \dots, W_{\pi_{k-1}}) \\ - H(Y_{\pi_1}^T, \dots, Y_{\pi_K}^T | W_{\pi_1}, \dots, W_K) + \epsilon_T \quad (255)$$

$$\leq \sum_{t=1}^T \left[\sum_{k=1}^K H(Y_{\pi_k t} | Y_{\pi_1 t}, \dots, Y_{\pi_{k-1} t}, X_{\pi_1 t}, \dots, X_{\pi_{k-1} t}) \right. \\ \left. - H(Y_{1t}, \dots, Y_{Kt} | X_{1t}, \dots, X_{Kt}) \right] + \epsilon_T \quad (256)$$

$$\leq T \max_{p(x_1, \dots, x_K)} \left[\sum_{k=1}^K H(Y_{\pi_k} | X_{\pi_1}, Y_{\pi_1}, \dots, X_{\pi_{k-1}}, Y_{\pi_{k-1}}) \right. \\ \left. - H(Y_1, \dots, Y_K | X_1, \dots, X_K) \right] + \epsilon_T, \quad (257)$$

where (252) follows from the independence of the messages, (253) follows from Fano's inequality [11], and (255) follows from the fact that the negative term corresponding to the k th bracket is cancelled by a part of the positive term in the $(k+1)$ th bracket, for $k = 1, \dots, (K-1)$. Finally, dividing (257) by T and letting $T \rightarrow \infty$, we have the proof of Theorem 4.

8.3 Proof of (147)

We first obtain a bound on the sum of the rates of users 1 and 2:

$$\begin{aligned} T(R_1 + R_2) &= H(W_1) + H(W_2) \end{aligned} \quad (258)$$

$$= H(W_1|W_3, \dots, W_K) + H(W_2|W_1, W_3, \dots, W_K) \quad (259)$$

$$\leq I(W_1; Y_1^T, Z_3^T, \dots, Z_K^T|W_3, \dots, W_K) + I(W_2; Y_2^T, Y_1^T, Z_3^T, \dots, Z_K^T|W_1, W_3, \dots, W_K) + \epsilon_T \quad (260)$$

$$\begin{aligned} &= h(Y_1^T, Z_3^T, \dots, Z_K^T|W_3, \dots, W_K) + h(Y_2^T|Y_1^T, Z_3^T, \dots, Z_K^T, W_1, W_3, \dots, W_K) \\ &\quad - h(Y_2^T, Y_1^T, Z_3^T, \dots, Z_K^T|W_1, W_2, W_3, \dots, W_K) + \epsilon_T \end{aligned} \quad (261)$$

$$\begin{aligned} &= h(Y_1^T, Z_3^T, \dots, Z_K^T|W_3, \dots, W_K) + h(Y_2^T|Y_1^T, Z_3^T, \dots, Z_K^T, W_1, W_3, \dots, W_K) \\ &\quad - \sum_{t=1}^T h(Y_{2t}, Y_{1t}, Z_{3t}, \dots, Z_{Kt}|W_1, W_2, W_3, \dots, W_K, Y_2^{t-1}, Y_1^{t-1}, Z_3^{t-1}, \dots, Z_K^{t-1}) + \epsilon_T \end{aligned} \quad (262)$$

$$\begin{aligned} &= h(Y_1^T, Z_3^T, \dots, Z_K^T|W_3, \dots, W_K) + h(Y_2^T|Y_1^T, Z_3^T, \dots, Z_K^T, W_1, W_3, \dots, W_K) \\ &\quad - \sum_{t=1}^T h(Y_{2t}, Y_{1t}, Z_{3t}, \dots, Z_{Kt}|X_{1t}, X_{2t}, X_{3t}, \dots, X_{Kt}) + \epsilon_T \end{aligned} \quad (263)$$

$$\begin{aligned} &= h(Y_1^T, Z_3^T, \dots, Z_K^T|W_3, \dots, W_K) + h(Y_2^T|Y_1^T, Z_3^T, \dots, Z_K^T, W_1, W_3, \dots, W_K) \\ &\quad - \sum_{t=1}^T h(Z_{1t}, Z_{2t}, Z_{3t}, \dots, Z_{Kt}) + \epsilon_T \end{aligned} \quad (264)$$

$$\begin{aligned} &\leq h(Y_1^T, Z_3^T, \dots, Z_K^T) + h(Y_2^T|Y_1^T, Z_3^T, \dots, Z_K^T, W_1, W_3, \dots, W_K) \\ &\quad - \sum_{t=1}^T h(Z_{1t}, Z_{2t}, Z_{3t}, \dots, Z_{Kt}) + \epsilon_T \end{aligned} \quad (265)$$

$$\begin{aligned} &\leq h(Y_1^T) + h(Z_3^T, \dots, Z_K^T) + h(Y_2^T|Y_1^T, Z_3^T, \dots, Z_K^T, W_1, W_3, \dots, W_K) \\ &\quad - \sum_{t=1}^T h(Z_{1t}, Z_{2t}, Z_{3t}, \dots, Z_{Kt}) + \epsilon_T \end{aligned} \quad (266)$$

$$\begin{aligned} &\leq TA + \sum_{t=1}^T h(Z_{3t}, \dots, Z_{Kt}) + h(Y_2^T|Y_1^T, Z_3^T, \dots, Z_K^T, W_1, W_3, \dots, W_K) \\ &\quad - \sum_{t=1}^T h(Z_{1t}, Z_{2t}, Z_{3t}, \dots, Z_{Kt}) + \epsilon_T \end{aligned} \quad (267)$$

$$\leq TA + \sum_{t=1}^T h(Y_{2t}|X_{1t}, Y_{1t}, X_{3t}) - \sum_{t=1}^T h(Z_{2t}) + \epsilon_T \quad (268)$$

$$= TA + \sum_{t=1}^T h(\sqrt{\text{SNR}}X_{2t} + Z_{2t}|X_{1t}, \sqrt{\text{INR}}X_{2t} + Z_{1t}, X_{3t}) - \sum_{t=1}^T h(Z_{2t}) + \epsilon_T \quad (269)$$

$$\leq TA + \sum_{t=1}^T h(\sqrt{\text{SNR}}X_{2t} + Z_{2t} | \sqrt{\text{INR}}X_{2t} + Z_{1t}) - \sum_{t=1}^T h(Z_{2t}) + \epsilon_T \quad (270)$$

$$\leq TA + T(C - E) + \epsilon_T \quad (271)$$

$$= T(A + C - E) + \epsilon_T, \quad (272)$$

where (259) follows from the independence of the messages, (260) follows from Fano's inequality, (262) follows from the chain rule, and (263) follows from the following argument:

$$\begin{aligned} X_{1t} &\text{ is a function of } (W_1, Y_1^{t-1}) \\ X_{2t} &\text{ is a function of } (W_2, Y_2^{t-1}), \\ X_{Kt} &\text{ is a function of } (W_K, X_1^{t-1}, Z_K^{t-1}), \\ X_{(K-1)t} &\text{ is a function of } (W_K, X_K^{t-1}, Z_{K-1}^{t-1}), \\ &\dots \\ X_{4t} &\text{ is a function of } (W_4, X_5^{t-1}, Z_4^{t-1}), \\ X_{3t} &\text{ is a function of } (W_3, X_4^{t-1}, Z_3^{t-1}). \end{aligned} \quad (273)$$

This argument allows us to write $(X_{1t}, X_{2t}, \dots, X_{Kt})$ in the conditioning in the last term in (263) and then use the memoryless property of the channel to arrive at (264).

The same argument also allows us to write (Y_{1t}, X_{1t}, X_{3t}) in the conditioning of the third term in (268) and subsequently drop all the remaining random variables from the conditioning. We remark here that this argument is similar to Claim 1 used in the proof of Theorem 3 for the linear deterministic model.

Finally, normalizing (272) by T and taking the limit $T \rightarrow \infty$, so that $\epsilon_T \rightarrow 0$, we have

$$R_1 + R_2 \leq A + C - E. \quad (274)$$

In a similar manner, it can be shown that for any $1 \leq j \leq K$, we have

$$R_j + R_{(j+1)} \leq A + C - E. \quad (275)$$

Adding all such K bounds, we obtain

$$2(R_1 + \dots + R_K) \leq K(A + C - E), \quad (276)$$

which yields

$$\mathcal{C}_{\text{sum,G}}^{\text{FB}}(K) \leq \frac{K}{2}(A + C - E). \quad (277)$$

Hence, we have proved the analogue of the type-I upper bound for the K -user Gaussian

References

- [1] O. Somekh, B. M. Zaidel, and S. Shamai (Shitz). Sum rate characterization of joint multiple cell-site processing. *IEEE Trans. on Information Theory*, 53(12):4473–4497, Dec. 2007.
- [2] A. D. Wyner. Shannon-theoretic approach to a Gaussian cellular multiple-access channel. *IEEE Trans. on Information Theory*, 40(6):1713–1727, Nov. 1994.
- [3] L. Zhou and W. Yu. On the capacity of the K-user cyclic Gaussian interference channel. *IEEE Trans. on Information Theory*, submitted, October 2010.
- [4] R. Etkin, D. Tse, and H. Wang. Gaussian interference channel capacity to within one bit. *IEEE Trans. on Information Theory*, 54(12):5534–5562, Dec. 2008.
- [5] Y. Liu and E. Erkip. On the sum capacity of K-user cascade Gaussian Z-interference channel. In *Proc. IEEE International Symposium on Information Theory (ISIT)*, St. Petersburg, Russia, 2011.
- [6] T. Han and K. Kobayashi. A new achievable rate region for the interference channel. *IEEE Trans. on Information Theory*, 27(1):49–60, January 1981.
- [7] G. Kramer. Feedback strategies for white Gaussian interference networks. *IEEE Trans. on Information Theory*, 48(6):1423–1438, June 2002.
- [8] M. Gastpar and G. Kramer. On noisy feedback for interference channels. In *Proc. Asilomar Conf. on Signals, Systems, and Computers*, Pacific Grove, CA, USA, Oct. 29–Nov. 1 2006.
- [9] R. Tandon and S. Ulukus. Dependence balance based outer bounds for Gaussian networks with cooperation and feedback. *IEEE Trans. on Information Theory*, 57(7):4063–4086, July 2011.
- [10] C. Suh and D. Tse. Feedback capacity of the Gaussian interference channel to within two bits. *IEEE Trans. on Information Theory*, 57(5):2667–2685, May 2011.
- [11] T. M. Cover and J. A. Thomas. *Elements of Information Theory*. New York: Wiley, 1991.